Math 8253 **Homework 5**
Posted: 3 a.m. 12/01/2017; Due: Monday, 12/11

The problem set is due at the beginning of the class on Monday, December 11.

Reading: **Class notes**. **Vakil**: Sections 2.2.H, 2.6–7, 4.4.4–5, 13.1–3, skipping 13.3.4, 13.4, 13.6, 16.1–3. **Hartshorne**: P. 65, Exercise II.1.18, Sections II.2.3.5–6, Exercise II.2.12, Section II.5 through 5.8. (All the exercises in this part of the assignment are for reference only; you do not have to solve them unless you want to.)

Problem 1. (1) Given an arbitrary point x in a topological space X and an abelian group F, define a presheaf $\mathcal{F}$ on X by

$$\mathcal{F}(U) := \begin{cases} F & \text{if } x \in U, \\ 0 & \text{otherwise} \end{cases}$$

for each open $U \subset X$. As in Homework 4 for a closed point x, it is easy to show that the presheaf $\mathcal{F}$ is a sheaf. You do not have to show that, as it is no different from the previous homework. The sheaf $\mathcal{F}$ is also called the *skyscraper sheaf* on X with stalk F at x, even though this is not necessarily the only nonzero stalk of $\mathcal{F}$. Show that $\mathcal{F}$ is the direct image $i_* F$ of the constant sheaf F on the one-point space $\{x\}$ under the inclusion $i : \{x\} \to X$.

(2) Now suppose that X is a scheme, $x \in X$ is a **closed** point, $k(x) = \mathcal{O}_{X,x}/\mathfrak{m}_x$ is its residue field, and F is a $k(x)$-vector space. Give the skyscraper sheaf $\mathcal{F}$ with stalk $\mathcal{F}_x = F$ an $\mathcal{O}_X$-module structure and show that it is quasi-coherent using Part (1) and the theorem on quasi-coherence of direct-image sheaves. [Note that $k(x)$ in this problem is the residue field of a point, whereas $k(t)$ on the previous homework was the field of rational functions of a formal variable t, i.e., the function field of the affine line $\mathbb{A}^1_k = \operatorname{Spec} k[t]$ over a field k.]

Problem 2. Show that the quasi-coherent skyscraper sheaf from Vakil's Exercise 13.2.A(b) is not locally of finite type and thereby not coherent (in the sense of our definition in class, which is equivalent to Vakil's Definition 13.6.4).

Problem 3. Show that the tensor product $\mathcal{F} \otimes_{\mathcal{O}_X} \mathcal{G}$ of quasi-coherent $\mathcal{O}_X$-modules $\mathcal{F}$ and $\mathcal{G}$ on a scheme X is quasi-coherent.

Problem 4. (1) Show that a scheme is quasi-compact iff it is a finite union of affine open subsets.

(2) Show the equivalence of the two definitions of a quasi-compact morphism of schemes in Vakil's 7.3.1.

Problem 5. Give an example of a morphism of schemes that is not quasi-compact. Give an example of a morphism of schemes that is not quasi-separated in the sense of Vakil's 7.3.1.

Problem 6. Let X be a topological space and $U \subset X$ an open subset. Let $j : U \hookrightarrow X$ be the inclusion. Show for any sheaf $\mathcal{F}$ of sets on X, that the inverse image $j^{-1}\mathcal{F}$ coincides with the restriction $\mathcal{F}|_U$ of $\mathcal{F}$ to U.

Problem 7. Exercise II.5.1(d) of Hartshorne.