MATH 8254: ALGEBRAIC GEOMETRY
PROBLEM SET 5, DUE FRIDAY, MAY 9, 2008

SASHA VORONOV

Problem 1. Hartshorne: II.6.1.

Problem 2. Hartshorne: III.7.3.

Problem 3. Let X be the complete nonsingular algebraic curve over an algebraically closed ground field k of characteristic $\operatorname{char} k \neq 2$ pasted from two nonsingular affine curves as follows. The first curve is given by the equation

$$y^2 = f(x)$$

in the xy plane, with the right-hand side

$$f(x) = a_d x^d + a_{d-1} x^{d-1} + \cdots + a_0$$

having d pairwise distinct roots $\alpha_1, \ldots, \alpha_d$. The second curve is defined by the equation

$$v^2 = u^{2m} f\left(\frac{1}{u}\right)$$

in the uv plane, where m is such that $d = 2m$ or $2m - 1$. These two affine curves are pasted together via $x = \frac{1}{u}$ and $y = \frac{v}{u^m}$. Note that this is not the curve given by the equation $y^2 = f(x)$ in $\mathbb{P}^2_k$, which was the one we talked about in class.

(1) Show that $g(X) = m - 1$. [Hint: This is what we actually did in class on Friday, May 2. Consider the rational form $\omega = \frac{dx}{y}$ and show it is regular and has two zeros of order $m - 2$ each, if $d = 2m$, and one zero of order $2m - 4$, if $d = 2m - 1$. Then recall that, for the corresponding divisor (ω), we have a formula for its degree: $\deg(\omega) = 2g - 2$.]

(2) Show that $\frac{dx}{y}, \frac{xdx}{y}, \ldots, \frac{x^{m-2}dx}{y}$ form a basis of the space $H^0(X, \Omega^1_{X/k})$ of regular 1-forms on X. [Hint: As in (1), show that each of these forms is regular. Thus, we have gotten $m - 1$ linearly independent regular 1-forms on X.]

Problem 4. If a divisor D on a complete nonsingular algebraic curve C satisfies $\deg D < 0$, prove that $H^0(C, \mathcal{O}_C(D)) = 0$.

Problem 5. For a complete nonsingular algebraic curve C over a field k, $k = \bar{k}$, and a closed point $P \in C$, put

$$A(P); = H^0(C, \mathcal{O}_C(*P)) := \{f \in k(C) \mid f \text{ is regular on } C\{P\}\}.$$

Show that $C \setminus \{P\} \cong \operatorname{Spec} A(P)$.

Date: May 3, 2008.