Math 8271 **Homework 6**

Posted: 11/25/2015; Updated: 11/26/2015; Due: Friday, 12/04/2015

The problem set is due at the beginning of the class on Friday, next week.

Reading: Sections 5.2-5.

Conventions: For this homework, $\mathfrak{g}$ is a finite-dimensional Lie algebra over a field $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$.

Problem 1. Complete checking the local diamond condition in the proof of the PBW theorem: two elementary reductions of $e_i e_j e_k$ with $i > j > k$ to $e_j e_i e_k + [e_i, e_j] e_k$ and $e_i e_k e_j + e_i [e_j, e_k]$ may be reduced further by similar (directed) moves to a common element of the tensor algebra $T\mathfrak{g}$.

Problem 2. Problem 5.7 from the textbook (Section 5.10). **Note**: You must assume that A is *strictly* upper-triangular. It is an unnoticed typo in the text.

Problem 3. Fix a natural number n and consider the operators of multiplication by $1, x, x^2, \dots, x^n$ and the operator $\partial/\partial x$ acting on the space of polynomials in x. Show that the linear span of these operators forms a nilpotent Lie algebra with respect to the commutator of linear operators.

Problem 4. Consider the operator of multiplication by e^x and the operator $\partial/\partial x$ acting on the space of smooth functions of one variable x. Show that the linear span of these operators forms a solvable Lie algebra with respect to the commutator of linear operators.

Problem 5. Show that every nilpotent Lie algebra has a nontrivial center.

Problem 6. Show that every two-dimensional nilpotent Lie algebra is Abelian.

Problem 7. Show that every three-dimensional nilpotent Lie algebra is either Abelian or defined by commutation relations $[p, q] = r$, $[p, r] = 0$, $[q, r] = 0$ in a basis $\{p, q, r\}$.