The problem set is due at the beginning of the class on Friday, April 15, 2016.

**Reading**: Text: Sections 7.9, 8.1–2.

**Problem from Chapter 7 (Section 7.11)**: 13.

**Problem from Chapter 8 (Section 8.10)**: 1, 3 [Hint: Solve Problem 1 below first], 9.

**Problem 1.** Let $P$ be the weight lattice of a polarized, reduced root system $R$ in a Euclidean space $E$ with a set $\{\alpha_1, \ldots, \alpha_r\}$ of simple roots. Let $\omega_i \in E$ be the fundamental weights, defined by

$\omega_i(\alpha_j^\vee) = \delta_{ij}, \quad i, j = 1, \ldots, r,$

where the (simple) coroots $\alpha_i^\vee \in E^*$ are defined by

$\lambda(\alpha_i^\vee) = \frac{2(\alpha_i, \lambda)}{\langle \alpha_i, \alpha_i \rangle}$

for all $\lambda \in E$.

Show that the weight lattice $P$ is the free Abelian group on the fundamental weights.

**Problem 2.** (1) Take the $B_n$ root system $R$ in the Euclidean space $\mathbb{R}^n$ with a standard basis $\{e_1, \ldots, e_n\}$, see page 205 (Appendix A.2). Show that its Dynkin diagram is indeed what it is supposed to be.

(2) Do the same for the $D_n$ root system from page 208, Appendix A.4. (Note that there is a misprint in the Dynkin diagram on page 208, see the Errata or page 153.)

**Problem 3.** Suppose that we have a connected Dynkin diagram corresponding to a reduced root system. Show that no more than three edges can be incident to a given vertex. Deduce that the only reduced root system with a connected Dynkin diagram and at least one triple edge is $G_2$. [You are not supposed to use the classification of Dynkin diagrams here. This is rather one of the steps in proving that theorem in the non simply-laced case.]

**Problem 4.** Fix a finite-dimensional complex semisimple Lie algebra. Prove a universality property of the Verma module $M_\lambda$ of highest weight $\lambda$ with a highest weight vector $v_\lambda$: given a highest weight representation $V$ of highest weight $\lambda$ with a highest-weight vector $v$, there is a unique morphism of representations $M_\lambda \to V$ mapping $v_\lambda$ to $v$. Show that this morphism will automatically be onto.