Here is brushed-up proof of duality between homology and cohomology with field coefficients.

**Theorem 1.** If $k$ is a field and $K$ a simplicial complex, then

$$H^n(K; k) = (H_n(K; k))^*$$

for all values of $n \geq 0$.

**Proof.** Step 1. For each $n \geq 0$, we have a natural isomorphism $C^n(K; k) = C_n(K; k)^*$ or $\text{Hom}_Z(C_n(K; \mathbb{Z}), k) = \text{Hom}_k(C_n(K; k), k)$, because $C_n(K; \mathbb{Z})$ is a free abelian group. It is freely generated by the set $K_n$ of $n$-simplices of $K$.

Step 2. Now we see that the cochain complex

$$\cdots \rightarrow C^{n-1}(K; k) \overset{\delta^n}{\rightarrow} C^n(K; k) \overset{\delta^{n+1}}{\rightarrow} C^{n+1}(K; k) \rightarrow \cdots$$

is a complex of vector spaces linear dual to the chain complex

$$\cdots \rightarrow C_{n+1}(K; k) \overset{\partial_{n+1}}{\rightarrow} C_n(K; k) \overset{\partial_n}{\rightarrow} C_{n-1}(K; k) \rightarrow \cdots$$

What remains to be shown is that the vector space $H^n(K; k) = \ker\delta^{n+1}/\im\delta^{n}$ is the linear dual of $H_n(K; k) = \ker\partial_n/\im\partial_{n+1}$.

Step 3. A remarkable thing about complexes of vector spaces is that given a short exact sequence (SES) $0 \rightarrow U \rightarrow V \rightarrow W \rightarrow 0$ of vector spaces, the linear dual sequence $0 \rightarrow W^* \rightarrow V^* \rightarrow U^* \rightarrow 0$ is also exact. This can be easily shown by *splitting* the given SES, that is to say, presenting $V$ as the direct sum of $U$ and a complementary subspace isomorphic to $W$, which gives $V \cong U \oplus W$ and thereby $V^* \cong U^* \oplus W^*$, yielding $0 \rightarrow W^* \rightarrow V^* \rightarrow U^* \rightarrow 0$. Such a complementary subspace always exists: complete a basis of $U$ to a basis of $V$, and take the linear span of the basis vectors which are not in $U$. This works even for infinite dimensional vector spaces, just requires the axiom of choice. Our simplicial complexes need not be finite: $K_n$, which forms a basis of $C_n(K; k)$, could well be an infinite set of simplices (*e.g.*, triangulate $\mathbb{R}^2$ by tiling it into triangles).

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Step 3 is where the argument would break, should we try to use it for abelian groups: dualize $0 \to \mathbb{Z} \xrightarrow{2\times} \mathbb{Z} \to \mathbb{Z}_2 \to 0$, that is to say, apply $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z})$ or $\text{Hom}_{\mathbb{Z}}(-, \mathbb{Z}_2)$ and see what happens. This is why there is no duality like that for homology and cohomology with arbitrary abelian coefficients.

Step 4. We have an SES

$$0 \to \text{Im} \partial_{n+1} \to \text{Ker} \partial_n \to H_n \to 0$$

This dualizes to an SES

$$0 \to H_n^* \to (\text{Ker} \partial_n)^* \to (\text{Im} \partial_{n+1})^* \to 0.$$  

Step 5: $(\text{Im} \partial_{n+1})^* = \text{Im} \delta^{n+1}$, because the linear map $C_{n+1} \xrightarrow{\partial_{n+1}} C_n$ can be factored into $\partial_{n+1} : C_{n+1} \xrightarrow{\partial_{n+1}} \text{Im} \partial_{n+1} \hookrightarrow C_n$, which dualizes to $\delta^{n+1} : C^n \twoheadrightarrow (\text{Im} \partial_{n+1})^* \xrightarrow{\delta^{n+1}} C^{n+1}$.

Step 6. $(\text{Ker} \partial_n)^* = C^n / \text{Im} \delta^n$, because the SES

$$0 \to \text{Ker} \partial_n \to C_n \to \text{Im} \partial_n \to 0$$

dualizes to an SES

$$0 \to (\text{Im} \partial_n)^* \to C^n \to (\text{Ker} \partial_n)^* \to 0$$

and $(\text{Im} \partial_n)^* = \text{Im} \delta^n$, as we have seen earlier (for any $n$).

Step 7. Collecting the last two steps, we get an SES

$$0 \to H_n^* \to C^n / \text{Im} \delta^n \to \text{Im} \delta^{n+1} \to 0$$

with the last linear map being induced by $\delta^{n+1}$. Thus, $H_n^*$ may be naturally identified with the kernel of this map, which is obviously $\text{Ker} \delta^{n+1} / \text{Im} \delta^n = H^n$. \qed