Math 8306 **Homework 2**

Posted: 9/23; Updated: 9/29; Due: Friday, 9/30/2016

The problem set is due at the beginning of the class on Friday, September 23.

Reading: Class notes. Text: Section 2.2 (pages 110, 158 (Exercise 32), 147–150), Section 3.1 (185–193 through the Exercise, 197–200, 195), Section 3.A (263). Beware that in the text these are done for singular homology, whereas we have been doing all that for simplicial homology.

Problem 1. Let K be a not necessarily connected simplicial complex in $\mathbb{R}^n$. Think of $\mathbb{R}^n$ as being triangulated, *i.e.*, provided with the structure of a simplicial complex, in such a way that K is a subcomplex. Then the exact sequence of a pair shows in particular that $H_1(\mathbb{R}^n, K; G) \cong \tilde{H}_0(K; G)$. A simplicial path in $\mathbb{R}^n$ between points in K will always be a relative 1-cycle. (No need to prove all that.) Show that such a path represents $0 \in H_1(\mathbb{R}^n, K; \mathbb{Z})$ if and only if the endpoints lie in the same component of K.

Problem 2. Think of the torus T^2 as the union of two cylinders glued along their boundary circles. Use the Mayer-Vietoris sequence to compute $H_\bullet(T^2; G)$. You may use homotopy invariance to identify the homology of the cylinder.

Problem 3. Suppose a pair of simplicial complexes $L \subset K$ is given and CL is the cone on the subcomplex L. Show that for all n we have isomorphisms $H_n(K, L; G) \cong H_n(K \cup CL, CL; G) \cong \tilde{H}_n(K \cup CL; G) \cong \tilde{H}_n(K/L; G)$.

Problem 4. Use the previous problem to compute the homology of $S^m \times S^n$ with coefficients in G for $m, n \geq 0$. (Assume $S^m \times S^n / S^m \vee S^n$ is homeomorphic to S^{m+n}, which will be easy to see when we will learn CW complexes.)

Problem 5. Let $K \subset S^3$ be a knot, a continuous embedding of S^1 into S^3. Calculate the integral homology $H_\bullet(S^3 \setminus K; \mathbb{Z})$ of the knot complement. Assume that the embedding of the knot extends to an embedding of the solid torus $S^1 \times \bar{D}^2$, where D^2 is the open unit disk in $\mathbb{R}^2$ and $\bar{D}^2$ is its closure. (Such knots are called *tame*.) Also assume you have a simplicial complex structure on $S^1 \times D^2$ and on the complement of the interior of the image of the solid torus in S^3, compatible on the intersection, which is the torus $S^1 \times S^1$ embedded in S^3.

Problem 6. Let K be a simplicial complex and $L_1, \dots, L_n$ be a collection of subcomplexes such that $K = L_1 \cup \dots \cup L_n$ and all nonempty intersections $L_{i_1 \dots i_p} := L_{i_1} \cap \dots \cap L_{i_p}$ are contractible. (Imagine covering a compact manifold by sufficiently small closed balls, *e.g.*, the circle by three arcs.) Define the *nerve* of the covering as the abstract simplicial complex N given by the set of subsets of $\{1, \dots, n\}$ defined as follows:

- a subset $\{i_1, \dots, i_p\} \subset \{1, \dots, n\}$ belongs to N iff $L_{i_1 \dots i_p}$ is nonempty.

(1) Take the barycentric subdivision K' of K and construct a simplicial map $f : K' \to N$ as follows. A vertex v in K' is the barycenter of a unique simplex in K. This simplex is in one of the L_i's. Define $f(v)$ to be the least such i. Check that this defines a simplicial map, *i.e.*, the image of each simplex in K' is a simplex in N.

(2) Show that $f_* : H_k(K'; G) \to H_k(N; G)$ is an isomorphism for all k. You may use the fact that $H_k(K'; G) \cong H_k(K; G)$. [*Hint*: Use induction on n and Mayer-Vietoris to complete the induction step.]

Problem 7. Let G be an abelian group and $nG := \{ng \mid g \in G\}$. Prove that $\mathrm{Hom}(G, \mathbb{Z}) \cong \mathrm{Hom}(nG, \mathbb{Z})$.