Math 8306 **Homework 3**

Posted: 10/04; Updated: 10/12; Due: Friday, 10/14/2016

The problem set is due at the beginning of the class on Friday, October 14.

Reading: Class notes. Text: Sections 2.2 (pages 137–141, 144, 153), 3.1 (190–196, 198), 3.3 (236–238, 241, 256), 3.A (261–266).

Conventions: No coefficients means integral coefficients: $H_n(K) := H_n(K; \mathbb{Z})$. A *closed manifold* is a smooth, compact manifold with no boundary.

Problem 1. Let

$$0 \to A \to B \to C \to 0$$

be a short exact sequence (SES) of abelian groups. Take a generating set of A and generate a free abelian group $F_0(A)$ by it. Complete the generating set of A to a generating set of B, call the resulting free abelian group $F_0(B)$. Let $F_0(C)$ be the free abelian group on the complementary generating set. We get epimorphisms $F_0(A) \to A$, $F_0(B) \to B$, and $F_0(C) \to C$. Let $F_1(A)$, $F_1(B)$, and $F_1(C)$ be their respective kernels. Prove by *diagram chasing* that the commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F_1(A) & \longrightarrow & F_1(B) & \longrightarrow & F_1(C) & \longrightarrow & 0 \\ & & \downarrow & & \downarrow & & \downarrow & & \\ 0 & \longrightarrow & F_0(A) & \longrightarrow & F_0(B) & \longrightarrow & F_0(C) & \longrightarrow & 0 \end{array}$$

has exact rows. *Note*: This is a quintessential problem on diagram chasing. We used this problem in class to deduce long exact sequences for Tor's and Ext's coming from an SES. As Tikhon pointed out, it also follows from the Snake Lemma, but I am asking you to use diagram chasing explicitly.

Problem 2. Let A be a finitely generated abelian group. Show that $\operatorname{Ext}(A, \mathbb{Z}) \cong A_T$, the *torsion subgroup* of A, *i.e.*, the "finite part" of A after applying the classification theorem to A. *Note*: There is another notation for the torsion subgroup: $\operatorname{Tor}(A) := A_T$.

Problem 3. Suppose an abelian group G is *divisible*, *i.e.*, $nG = G$ for each positive integer n. Prove that $\operatorname{Ext}(A, G) = 0$ for any abelian A. *Note*: The groups $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, and $\mathbb{Q}/\mathbb{Z}$ are obviously divisible.

Problem 4. Suppose K is a simplicial complex with a finitely generated group $H_1(K)$. Decompose this group into free and torsion parts $H_1(K) \cong \mathbb{Z}^r \oplus T$. Compute $H^1(K)$.

Problem 5. Let $K \to L$ be a continuous map between finite simplicial complexes. Suppose that $f_* : H_1(K) \to H_1(L)$ is zero. Show that so is $f^* : H^1(L) \to H^1(K)$.

Problem 6. Suppose M^n is a connected closed orientable manifold. Prove that $H_{n-1}(M^n \setminus D^n) \cong H_{n-1}(M^n)$, where D^n is an open ball in a chart of M^n.

Problem 7. Compute the integral homology of an n-torus $T^n = \mathbb{R}^n/\mathbb{Z}^n$ for $n \geq 0$, using cellular homology.

Problem 8. Suppose M^n is a closed orientable manifold and its (unreduced) suspension $SM = M \times I/\{M \times \{0\}\}/\{M \times \{1\}\}$ is homeomorphic to a closed orientable manifold. Prove that M is a *homology sphere*, *i.e.*, has the same integral homology as S^n.