Problem 1. Give an example of two CW complexes which have the same homology but are not homotopy equivalent.

Problem 2. Show that every self-map of $\mathbb{R}P^n$ has a fixed point.

Problem 3. Let $A$ be a subcomplex of a CW complex $X$, $Y$ a CW complex, $f: A \to Y$ a cellular map, and $Y \cup_f X$ the pushout (i.e., the attaching space). The Euler characteristic of a finite CW complex $X$ may be defined as $\chi(X) = \sum_{n} (-1)^n \gamma_n(X)$. Formulate and prove a formula relating the Euler characteristics of $A$, $X$, $Y$, and $X \cup_f Y$, when $X$ and $Y$ are finite.

Problem 4. Let $p: X \to Y$ be a covering map with finite fibers of cardinality $n$. Using singular chains, construct a homomorphism $\tau: H_\bullet(Y; G) \to H_\bullet(X; G)$ such that the composite $p^* \circ \tau: H_\bullet(Y; G) \to H_\bullet(Y; G)$ is a multiplication by $n$. [This is called a transfer homomorphism.]

Problem 5. Using cell decompositions of $S^n$ and $D^n$, compute their cellular homology groups.

Problem 6. Calculate the integral homology groups of the sphere $M_g$ with $g$ handles (i.e., a compact orientable surface of genus $g$) using a cell decomposition of it.

Problem 7. Prove the “coassociativity” property $(\text{id} \otimes \text{AW}) \circ \text{AW} = (\text{AW} \otimes \text{id}) \circ \text{AW}$ of the Alexander-Whitney map $\text{AW}: S_\bullet(X \times Y) \to S_\bullet(X) \otimes S_\bullet(Y)$.

Problem 8. Give a simple proof that uses homology theory for the following fact: $\mathbb{R}^m$ is not homeomorphic to $\mathbb{R}^n$ for $m \neq n$.

Problem 9. Regarding a singular cochain $\phi \in C^1(X; G)$ as a function from paths in $X$ to $G$, show that if $\phi$ is a cocycle, that is, $\delta \phi = 0$, then

1. $\phi(f \cdot g) = \phi(f) + \phi(g)$,
2. $\phi$ takes the value 0 on constant paths,
3. $\phi(f) = \phi(g)$ if $f \sim g$ via a homotopy fixing the endpoints,
4. $\phi$ is a coboundary (that is, $\phi = \delta \psi$ for some $\psi \in C^0(X; G)$) iff $\phi(f)$ depends only on the endpoints of $f$, for all $f$.

[In particular, 1 and 4 give a homomorphism $H^1(X; G) \to \text{Hom}(\pi_1(X), G)$, which is a version of Hurewicz isomorphism if $X$ is path connected.]

Problem 10. Assuming as known the cup product structure on the torus $T^2 = S^1 \times S^1$, $(\langle e^0 \rangle^* \cup a = a$ for a generator $\langle e^0 \rangle^*$ of $H^0(T; \mathbb{Z}) = \mathbb{Z}$ and any $a \in H^\bullet(T; \mathbb{Z})$, $\langle e^1 \rangle^* \cup \langle e^1 \rangle^* = (e^2)^*$ and $\langle e^1 \rangle^* \cup \langle e^1 \rangle^* = 0$ for $i = 1, 2$ and generators $\langle e^1 \rangle^*$ and
$\langle e_1 \rangle^* \ast H^1(T; \mathbb{Z})$ coming from two natural projections $T \to S^1$ via pullback, and $\langle e_2 \rangle^* \cup \langle e_1 \rangle^* = \langle e_2 \rangle^* \cup \langle e_2 \rangle^* = 0$, compute the cup product in $H^\bullet(M_g; \mathbb{Z})$ for the compact orientable surface $M_g$ of genus $g$, by using a quotient map from $M_g$ to the bouquet of $g$ tori.

Problem 11. Show that any map $S^4 \to S^2 \times S^2$ must induce the zero homomorphism on $H_4(-)$. [Hint: Use the cup product]

Problem 12. Prove that there is no homotopy equivalence $f : \mathbb{C}P^{2n} \to \mathbb{C}P^{2n}$ that reverses orientation (induces multiplication by $-1$ on $H_{4n}(\mathbb{C}P^{2n}; \mathbb{Z})$). [Hint: note from cellular cohomology that the generator $\langle e^{4n} \rangle^* \in H^{4n}$ is a cup square $\langle e^{2n} \rangle^* \cup \langle e^{2n} \rangle^*$, which you may assume is true. To pass from cohomology to homology, use the naturality of the Universal Coefficients Theorem.]