MATH 8307: ALGEBRAIC TOPOLOGY
PROBLEM SET 5, DUE APRIL 7, 2003

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Problem 1. Determine the integral homology of the total space of a fibration over $S^2$ with fiber $S^3$.

Problem 2. Consider the cohomology spectral sequence of a three-sphere bundle $U(2)/U(1) = S^3 \to BU(1) \to BU(2)$ with coefficients in $\mathbb{R}$. Show that $E^2_{p,3} = E^p_{3}$ and $E^p_{2,4} = E^p_{2,4}$. Using $H^*\left(BU(1); \mathbb{R}\right) \cong \mathbb{R}[c_1]$, $c_1 \in H^2\left(BU(1); \mathbb{R}\right)$, show that $d_4 : E^2_{p,3} \to E^p_{2,4}$ is injective for $p \geq 0$.

Problem 3. For the Leray-Serre cohomology spectral sequence of a fibration $S^n \to E \xrightarrow{p} B$ over a simply connected base $B$, show that there exists an element $\Omega \in H^{n+1}(B; \mathbb{R})$, such that the map $\Psi : H^p(B; \mathbb{R}) \to H^{p+n+1}(B; \mathbb{R})$ defined by $\Psi(u) = u \cup \Omega$, $u \in H^p(B; \mathbb{R})$, induces the following exact sequence

$$\cdots \to H^{p+n}(E; \mathbb{R}) \to H^p(B; \mathbb{R}) \xrightarrow{\Psi} H^{p+n+1}(B; \mathbb{R}) \xrightarrow{\tau} H^{p+n+1}(E; \mathbb{R}) \to \cdots$$

This is called the cohomology Gysin sequence of a sphere fibration.

Problem 4. Show that for $n \geq 1$

$$H_*(\Omega^2S^{2n+1}; \mathbb{Z}_p) \cong \bigotimes_{k=0}^{\infty} \mathbb{Z}_p[u_{2kp^k-1}] \otimes \bigotimes_{k=1}^{\infty} \mathbb{Z}_p[v_{2np^{k-2}}]$$

as a $\mathbb{Z}_p$-vector space, where $p$ is a prime and $\mathbb{Z}_p[x_m]$ denotes the graded symmetric algebra on an element of degree $m$. (So that when $m$ is even, we have a polynomial algebra, and when $m$ is odd, we have an exterior algebra). [Hint: Use the cohomological Leray-Serre spectral sequence for the fibration $\Omega^2S^{2n+1} \to P\Omega S^{2n+1} \to \Omega S^{2n+1}$ and the computation of the cohomology ring $H^*\left(\Omega S^{2n+1}; \mathbb{Z}\right)$ as the divided polynomial ring $\mathbb{Z} \oplus \mathbb{Z}x \oplus \cdots \oplus \mathbb{Z}x^i/1! \oplus \cdots$ in an element $x \in H^{2n}(\Omega S^{2n+1}; \mathbb{Z})$, see p. 224 of Hatcher, where the cohomology ring of the James construction $J(S^{2n}) \sim \Omega\Sigma S^{2n} \sim \Omega S^{2n+1}$ is computed. To compute the first nonzero differential $d_{2n}$, note that $d_{2n} : E^{2n-1}_{2,2n} \to E^2_{2,0}$ must be an isomorphism, which implies there exists an element $u \in H^{2n-1}(\Omega^2S^{2n+1}; \mathbb{Z}_p)$ such that $d_{2n}(1 \otimes u) = x \otimes 1$. Use the derivation property to compute $d_{2n}(x^i/1!) \otimes u)$. Show that the basis $\{1, u\}$ is a basis of $H^*\left(\Omega^2S^{2n+1}; \mathbb{Z}_p\right)$ in dimensions $< 2pn - 2$. Proceed in this fashion. See that this continues to the right for $p$ steps, then breaks, creating a new pattern, which continues to the right for $p$ steps, creating another pattern, and so on. After you have computed the cohomology, use the universal coefficients formula to compute the homology.]

Problem 5. Consider an Eilenberg-Mac Lane space $K(\mathbb{Z}, 3)$. Let $j : S^3 \to K(\mathbb{Z}, 3)$ be a generator of $\pi_3(K(\mathbb{Z}, 3))$ and $S(3)$ denote the homotopy fiber of $j$, i.e., the

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fiber of the mapping path fibration of $j$. Consider the fiber sequence
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\cdots \rightarrow \Omega K(\mathbb{Z}, 3) \rightarrow S^3(3) \rightarrow S^3 \rightarrow K(\mathbb{Z}, 3).
\]
How do the homotopy groups of $S^3(3)$ compare to those of $S^3$? Note that $\Omega K(\mathbb{Z}, 3) = K(\mathbb{Z}, 2) = \mathbb{C}P^\infty$ up to homotopy equivalence. Use the Leray-Serre spectral sequence for the homotopy fibration $\mathbb{C}P^\infty \rightarrow S^3(3) \rightarrow S^3$ to calculate $H_\bullet(S^3(3))$ and use the result to show that $\pi_4(S^3) \cong \mathbb{Z}/2\mathbb{Z}$. [Hint: Use the cohomological Leray-Serre spectral sequence and the derivation property to compute $d_3$ explicitly, using the cohomology ring of $\mathbb{C}P^\infty$. Then go back to homology by UCT.]

**Problem 6.** Read the definition of a Thom class and the Thom space of a disk bundle in Hatcher, p. 441 of a hardcopy. Prove the Thom isomorphism theorem, Corollary 4D.9, using the Leray-Serre spectral sequence for an $S^n$-bundle you associate with the $D^n$-bundle (neither the Leray-Hirsch theorem, which is Theorem 4D.8, nor the Mayer-Vietoris sequence).