MATH 8307: ALGEBRAIC TOPOLOGY
PROBLEM SET 5, DUE FRIDAY, APRIL 15, 2005

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From Hatcher’s textbook:
Section 4.1: 18
Section 4.2 (from Hatcher’s textbook): 15 (In fact, prove a more general statement that a closed connected simply-connected \( n \)-dimensional manifold whose homology is the same as that of \( S^n \) is homotopy equivalent to \( S^n \).) [Hint: use the Hurewicz theorem.]

Problem 1. Show that \( \pi_7(S^4) \) contains a \( \mathbb{Z} \) summand.

Problem 2. Compute all the homotopy groups of \( \mathbb{R}P^\infty = \bigcup_{n \geq 1} \mathbb{R}P^n \), using the computation of the homotopy groups of \( \mathbb{R}P^n \) through the homotopy groups of spheres \( S^n \).

Problem 3. Regarding a singular cochain \( \phi \in C^1(X; G) \) as a function from paths in \( X \) to \( G \), show that if \( \phi \) is a cocycle, that is, \( \delta \phi = 0 \), then

1. \( \phi(f \cdot g) = \phi(f) + \phi(g) \),
2. \( \phi \) takes the value 0 on constant paths,
3. \( \phi(f) = \phi(g) \) if \( f \sim g \) via a homotopy fixing the endpoints,
4. \( \phi \) is a coboundary (that is, \( \phi = \delta \psi \) for some \( \psi \in C^0(X; G) \)) iff \( \phi(f) \) depends only on the endpoints of \( f \), for all \( f \).

[In particular, 1 and 4 give a homomorphism \( H^1(X; G) \to \text{Hom}(\pi_1(X), G) \), which is a version of Hurewicz isomorphism if \( X \) is path connected.]

Problem 4. Show that, if \( n \geq 2 \), then \( \pi_n(X \vee Y) \) is isomorphic to

\[ \pi_n(X) \oplus \pi_n(Y) \oplus \pi_{n+1}(X \times Y, X \vee Y). \]

Problem 5. Compute \( \pi_n(\mathbb{R}P^n, \mathbb{R}P^{n-1}) \) for \( n \geq 2 \). Deduce that the quotient map

\[ (\mathbb{R}P^n, \mathbb{R}P^{n-1}) \to (\mathbb{R}P^n / \mathbb{R}P^{n-1}, *) \]

does not induce an isomorphism of homotopy groups.

Problem 6. Assume given maps \( f : X \to Y \) and \( g : Y \to X \) such that \( g \circ f \simeq \text{id}_X \). Suppose that \( Y \) is a CW complex. Show that \( X \) has the homotopy type of a CW complex, i.e., is homotopy equivalent to a CW complex.

Problem 7. Let \( n \geq 1 \) and \( \pi \) be an abelian group. Construct a connected CW complex \( X \) such that \( \tilde{H}_n(X; \mathbb{Z}) = \pi \) and \( \tilde{H}_q(X; \mathbb{Z}) = 0 \) for \( q \neq n \). Such space \( X \) is denoted \( M(\pi, n) \) and called a Moore space. [Hint: construct \( M(\pi, n) \) as the cofiber of a map between wedges of spheres.]

Date: March 31, 2005.
Problem 8. Let $n \geq 1$ and $\pi$ be an abelian group. Construct a connected CW complex $X$ such that $\pi_n(X) = \pi$ and $\pi_q(X) = 0$ for $q \neq n$. Such space $X$ is denoted $K(\pi, n)$ and called an Eilenberg-Mac Lane space. [Hint: start with $M(\pi, n)$, use the Hurewicz theorem, and kill the higher homotopy groups.]