LECTURE 2: PROP'S

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1. PROP's

At this point, we are ready to introduce the notion of a PROP, see *e.g.*, J. Adams' book on infinite loop spaces or MacLane's original paper [**?**], the big brother of an operad, so that it would be unwise not to mention it here.

Definition 1. A *PRO* is a symmetric monoidal category whose objects are the nonnegative integers with the tensor product given by

$$m \otimes n = m + n.$$

It is not by mistake that the letter "P" in "PROP" is missing. The matter is that PROP is supposed to stand for PROducts, *i.e.*, the compositions of morphisms in the category, and Permutations. Thus, a PROP is a PRO with some permutation data mixed in. More precisely,

Definition 2. A *PROP* is a symmetric monoidal (sometimes called tensor) category whose set of objects is identified with the set $\mathbb{Z}_+$ of nonnegative integers, where the tensor law on $\mathbb{Z}_+$ is given by addition and the associativity transformation α is equal to identity. See the founding fathers' sources, such as, J. F. Adams' book [**?**] or S. Mac Lane's paper [**?**] for more detail.

Example 3 (The Segal PROP)**.** The *Segal PROP* is a PROP of infinite dimensional complex orbifolds. The space of morphisms is defined as the moduli space $\mathcal{P}_{m,n}$ of complex Riemann surfaces bounding $m+n$ labeled nonoverlapping holomorphic holes. The surfaces should be understood as compact smooth complex curves, not necessarily connected, along with $m+n$ biholomorphic maps of the closed unit disk to the surface. The more exact nonoverlapping condition is that the closed disks in the inputs do not intersect pairwise and the closed disks in the outputs do not intersect pairwise, however, an input and an output disk may have common boundary, but are still not allowed to intersect at an interior point. This technicality brings in the symmetric group morphisms, including the identity, to the PROP, but does not create singular Riemann surfaces by composition. The moduli space means that we consider isomorphism classes of such objects. The composition of morphisms in this PROP is given by sewing the Riemann surfaces along the boundaries, using the equation $zw = 1$ in the holomorphic parameters coming from the standard one on the unit disk. The tensor product of morphisms is the disjoint union. This PROP plays a crucial role in Conformal Field Theory, as we will see in the next lecture.

Example 4. Another example is the *endomorphism PROP of a vector space* V: the set of morphisms from m to n is defined as $\mathrm{Mor}(m, n) = \mathrm{Hom}(V^{\otimes m}, V^{\otimes n})$.

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A PROP is the big brother of an operad: an *operad* is basically the part $\mathrm{Mor}(n, 1)$ of a PROP. What this means will be the topic of a coming lecture.