HIGHER OPERATIONS ON HOCHSCHILD COMPLEX

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A recent explosion of algebraic structures derived in quantum field theory [14, 15] and in the theory of vertex operator algebras [13] has led to the renaissance of operads and algebras with several operations. The physicists' vision of the Universe revealed evidence of a number of new mathematical structures now being intensively studied by mathematicians, [6, 7, 8, 9, 11, 13]. Comparing the two results:

(1) (Gerstenhaber, [4]) The Hochschild cohomology $\mathrm{HH}^\bullet(A) := H^\bullet(A, A)$ of an associative algebra A has a natural structure of a *G algebra*, short for a *Gerstenhaber algebra*.

(2) (F. Cohen, [2]) An algebra over the homology little squares operad is nothing but a G algebra.

and, perhaps, the new insights on homotopy algebras [8, 12], P. Deligne posed the following conjecture in his letter [3].

Conjecture 1 (Deligne)**.** *The Hochschild cochain complex* $\mathcal{C}^\bullet(A) := C^\bullet(A, A)$ *has a natural structure of an algebra over a chain operad of the little squares operad.*

A "chain operad" here may mean any operad of complexes, i.e., a *DG operad*, whose homology operad is isomorphic to the homology of the little squares operad. In view of Huang's work [9], this conjecture suggests that *the Hochschild cochain complex* $\mathcal{C}^\bullet(A)$ *has a structure similar to that of a topological vertex* (*operator*) *algebra.*

Since a DG operad whose homology operad describes (associative, commutative, Lie, etc.) algebras is supposed to describe homotopy (associative, commutative, Lie, etc., respectively) algebras, Conjecture 1 can also be treated as follows.

Conjecture 2. *The Hochschild cochain complex* $\mathcal{C}^\bullet(A)$ *has a natural structure of a homotopy G algebra.*

In this paper, we present the proof of Conjecture 2, see Theorem 3. A "homotopy G algebra" in the conjecture may mean any type of homotopy algebras whose operations induce the structure of a G algebra

Research of the first author was supported in part by an NSA grant.

Research of the second author was supported in part by NSF grant DMS-9108269.A03.

on cohomology. The moral reason of the amazing connection between Hochschild cohomology and the topology of little squares is still a mystery. A plausible explanation is suggested in Section 11.

1. **The Hochschild complex.** Let A be an associative algebra over a field of characteristic zero and

$$V = \mathcal{C}^\bullet(A) = C^\bullet(A, A)$$

be the Hochschild complex of A, computing the Hochschild cohomology of A with coefficients in itself. We will consider the following operations on the Hochschild complex, denoting elements of $V = \mathcal{C}^\bullet(A)$ by x, x_1, etc., and elements of A by a_1, a_2, etc. Let also $|x|$ denote the degree of an element x in the desuspension of V:

$$|x| = k - 1 \text{ if } x \in \mathcal{C}^k(A).$$

In particular, we assume that for $a \in A = \mathcal{C}^0(A)$, $|a| = -1$.

2. **The dot (cup) product.** For cochains $x \in \mathcal{C}^k(A)$ and $y \in \mathcal{C}^l(A)$, define

$$(x \cdot y)(a_1, \dots, a_{k+l}) := (-1)^{(|x|+1)(|y|+1)} x(a_1, \dots, a_k) \cdot y(a_{k+1}, \dots, a_{k+l}). \tag{1}$$

This is the usual cup product $x \cup y$ of cochains altered by the sign. *The sign convention is that we insert the sign* $(-1)^{|x||y|}$ *when an object* x *passes through an object* y*, assuming the degree of "$\cdot$" is one, the degree of* d (*see below*) *is one, etc.* It is easy to verify that $xy = x \cdot y$ defines the structure of an associative differential graded (DG) algebra on V, where the differential

$$\begin{aligned}(dx)(a_1, \dots, a_{n+1}) := {} & (-1)^{|x|} a_1 x(a_2, \dots, a_{n+1}) \\ & + (-1)^{|x|} \sum_{i=1}^{n} (-1)^i x(a_1, \dots, a_{i-1}, a_i a_{i+1}, a_{i+2}, \dots, a_{n+1}) \\ & + x(a_1, \dots, a_n) a_{n+1}\end{aligned}$$

in the Hochschild complex plays the role of a derivation:

$$d(x \cdot y) = dx \cdot y + (-1)^{|x|+1} x \cdot dy. \tag{2}$$

This algebra is differential graded in the sense of the usual degree $|x|+1$.

3. **The braces.**

$$
\begin{aligned}
&x\{x_1, \dots, x_m\}(a_1, \dots, a_n) \\
:= &\sum_{0 \le i_1 \le \dots \le i_m \le n} (-1)^\varepsilon x(a_1, \dots, a_{i_1}, x_1(a_{i_1+1}, \dots), \dots, a_{i_m}, x_m(a_{i_m+1}, \dots), \dots, a_n),
\end{aligned}
\tag{3}
$$

where $\varepsilon := \sum_{p=1}^m |x_p| i_p$, i.e., $(-1)^\varepsilon$ is the sign picked up by rearranging the sequence of letters on the left-hand side of the formula into the sequence on the right-hand side, in accordance with the usual sign convention. The braces $x\{x_1, \dots, x_m\}$ are homogeneous of degree zero with respect to the suspended degree $|x|$. It will be convenient to denote $x\{y\}$ also by $x \circ y$ and understand $x\{\}$ as just x.

Let $m \in \mathcal{C}^2(A)$, $m(a \otimes b) = ab$, be the multiplication cocycle. It can be checked directly that

$$x \cdot y = (-1)^{|x|+1} m\{x, y\}, \tag{4}$$

$$dx = m \circ x - (-1)^{|x|} x \circ m. \tag{5}$$

4. **Identities.** The operations introduced above satisfy a nice collection of identities, which we are going to list. First of all, V is an associative DG algebra with respect to the dot product and the degree $|x| + 1$. Apart from that, there are the following more sophisticated identities.

5. **Higher pre-Jacobi identities.**

$$
\begin{aligned}
&x\{x_1, \dots, x_m\}\{y_1, \dots, y_n\} \\
= &\sum_{0 \le i_1 \le \dots \le i_m \le n} (-1)^\varepsilon x\{y_1, \dots, y_{i_1}, x_1\{y_{i_1+1}, \dots\}, \dots, y_{i_m}, x_m\{y_{i_m+1}, \dots\}, \dots, y_n\},
\end{aligned}
\tag{6}
$$

where $\varepsilon := \sum_{p=1}^m |x_p| \sum_{q=1}^{i_p} |y_q|$, i.e., the sign is picked up by the x_i's passing through the y_j's in the shuffle. This is a straightforward computation.

6. **Distributivity.**

$$(x_1 \cdot x_2)\{y_1, \dots, y_n\} = \sum_{k=0}^n (-1)^\varepsilon x_1\{y_1, \dots, y_k\} \cdot x_2\{y_{k+1}, \dots, y_n\}, \tag{7}$$

where $\varepsilon = (|x_2| + 1) \sum_{p=1}^k |y_p|$. In view of (4), this is a particular case of (6).

7. **Higher homotopies.**

$$\begin{aligned} d(x\{x_1,\dots,x_{n+1}\}) &- (dx)\{x_1,\dots,x_{n+1}\} \\ - (-1)^{|x|}\sum_{i=1}^{n+1} &(-1)^{|x_1|+\dots+|x_{i-1}|} x\{x_1,\dots,dx_i,\dots,x_{n+1}\} \\ &= (-1)^{(|x|+1)|x_1|} x_1\cdot x\{x_2,\dots,x_{n+1}\} \\ &\quad - (-1)^{|x|}\sum_{i=1}^{n}(-1)^{|x_1|+\dots+|x_i|} x\{x_1,\dots,x_i\cdot x_{i+1},\dots,x_{n+1}\} \\ &\quad + (-1)^{|x|+|x_1|+\dots+|x_n|} x\{x_1,\dots,x_n\}\cdot x_{n+1} \end{aligned} \tag{8}$$

This is also a particular case of (6), because of (4) and (5).

8. **Homotopy G algebras.** A *homotopy G algebra* is a complex $V = \bigoplus_n V^n$ of vector spaces with a differential d of degree one, with a dot product xy making the suspension $V[-1]$, $V[-1]^n := V^{n-1}$, into an associative DG algebra and with a collection of braces $x\{x_1,\dots,x_n\}$, $n \geq 0$, satisfying the identities (6), (7) and (8). We imply that the dot product has degree one with respect to the genuine grading of V and that the braces have degree zero: $|xy| = |x| + |y| + 1$ and $|x\{x_1,\dots,x_n\}| = |x|+|x_1|+\dots+|x_n|$ if the degree of $x \in V^n$ is denoted by $|x|$. Sections 4–7 can be summarized as the following result.

Theorem 3. *The desuspension $V = \mathcal{C}^\bullet(A)[1]$ of the Hochschild complex of an associative algebra A with the dot product* (1) *and the braces* (3) *is a homotopy G algebra.*

9. **Identities for some lower braces.** Equation (6) for $m = n = 1$ implies the *pre-Jacobi identity*:

$$(x\circ y)\circ z - x\circ(y\circ z) = (-1)^{|y||z|}((x\circ z)\circ y - x\circ(z\circ y)). \tag{9}$$

For $n = 1$, (7) turns into the *right Leibniz rule*:

$$(xy)\circ z = (-1)^{(|y|+1)|z|}(x\circ z)y + x(y\circ z). \tag{10}$$

Specifying $n = 0$ in (8), we get the *homotopy commutativity*:

$$xy - (-1)^{(|x|+1)(|y|+1)}yx = (-1)^{|x|}(d(x\circ y) - dx\circ y - (-1)^{|x|}x\circ dy). \tag{11}$$

Finally, putting $n = 1$ in the same (8), we get the *homotopy left Leibniz rule*:

$$\begin{aligned} &x\circ(yz) - (x\circ y)z - (-1)^{|x|(|y|+1)}y(x\circ z) \\ &= (-1)^{|x|+|y|+1}(d(x\{y,z\}) - (dx)\{y,z\} - (-1)^{|x|}x\{dy,z\} - (-1)^{|x|+|y|}x\{y,dz\}). \end{aligned} \tag{12}$$

10. **G algebras.** A *G algebra* is a graded vector space H with a dot product xy defining the structure of a graded commutative algebra on the suspension $H[-1]$ and with a bracket $[x, y]$ defining the structure of a graded Lie algebra on H, such that the bracket with an element is a derivation of the dot product:

$$[x, yz] = [x, y]z + (-1)^{|x|(|y|+1)} y[x, z].$$

In other words, a G algebra is a specific graded version of a Poisson algebra.

Corollary 4. *The dot product and the bracket* $[x, y] := x \circ y - (-1)^{|x||y|} y \circ x$ *define the structure of a G algebra on the Hochschild cohomology* $\mathrm{HH}^\bullet(A)$ *of an associative algebra* A.

Proof. A simple computation shows that the pre-Jacobi identity (9) yields the Jacobi identity for the bracket. Equation (11) implies that the differential is a derivation of the bracket:

$$d[x, y] - [dx, y] - (-1)^{|x|}[x, dy] = 0. \tag{13}$$

Therefore, even before passing to cohomology, the Hochschild complex forms a DG Lie algebra with respect to the bracket and an associative DG algebra with respect to the dot product.

Thus, we will be through if we see that

(1) the two operations take cocycles into cocycles and are independent of the choice of representatives of cohomology classes,
(2) the dot product is graded commutative and
(3) the bracket is a derivation of the dot product.

It easy to observe (1) from (2) and (13), (2) from the homotopy commutativity (11) at the cochain level and (3) from the right Leibniz and the homotopy left Leibniz rules (10) and (12). □

Remark 1. The structure of G algebra on Hochschild cohomology $\mathrm{HH}^\bullet(A)$ was introduced by Gerstenhaber in [4]. Equations (9)–(12) can also be found in that paper. The bracket $[z, z]$ plays the role of a primary obstruction in deformation theory. The braces (3) have been introduced in Getzler's work [5], where they are used to define the Hochschild cohomology of a homotopy associative algebra.

11. **Open problems.** There is a canonical definition of a homotopy G algebra as an algebra over the operad cobar-dual to the homology little squares operad, see [8], but our notion of a homotopy G algebra is different from that one — it contains fewer homotopies for the dot product and more homotopies for the bracket. In fact, our notion does not fit the general scheme of quadratic operad theory [8]. Firstly, we

have used the circle product $x \circ y$, which is a bilinear homotopy, see (11), but in quadratic operad theory, no bilinear homotopies are allowed. This is because the commutativity law is regarded in operad theory as a linear relation (it is really linear in structure constants), but for quadratic operads the only admissible relations are quadratic. Moreover, relations (6) and (8) contain not only quadratic (corresponding to compositions of two operations) but linear terms. Thus, Koszul duality for quadratic operads [8] must be adjusted to treat linear relations as well as quadratic. Secondly, relations (7) contain cubic as well as quadratic terms. However, it does not mean that the Hochschild complex fails to have the structure of a homotopy G algebra in the sense of (perhaps, generalized) quadratic operad theory:

Problem 5. *Does the Hochschild complex of an associative algebra have a natural structure of an algebra over the operad cobar-dual to the operad describing G algebras*?

We believe that our proof of Conjecture 2 provides more evidence for Conjecture 1. Perhaps, there is more evidence for the following particular case of this conjecture.

Problem 6. *When $A = \Omega^\bullet(X)$ is the ring of differential forms on a manifold X, the Hochschild complex of A is an algebra over the operad of chains in the operad of Riemann spheres with nonoverlapping holomorphic disks.*

This problem is related to Conjecture 1 and also supported by the following speculation. Due to the results of Chen [1] and Jones [10], the Hochschild complex $\mathcal{C}^\bullet(A)$ is naturally quasi-isomorphic to the singular chain complex $C_\bullet(\mathcal{LX})$ of the free loop space $\mathcal{LX}$. Thus the hypothetic structure of Problem 6 may invoke the same on $C_\bullet(\mathcal{LX})$. On the other hand, on the space of semi-infinite forms $\Omega^{\infty/2+\bullet}(\mathcal{LX})$, which is supposed to be intermediate between $\Omega^\bullet(\mathcal{LX})$ and $C_\bullet(\mathcal{LX})$, this structure is readily given by sigma-model, where $\Omega^{\infty/2+\bullet}(\mathcal{LX})$ plays the role of the state space. Therefore, one may need to replace the Hochschild complex of A in the problem by some semi-infinite version of it. Then the associative homotopy commutative product on $\Omega^{\infty/2+\bullet}(\mathcal{LX})$ will be the quantum multiplication of mirror symmetry. Note also, that according to Lian and Zuckerman [13], there is a structure of a G algebra on the semi-infinite cohomology of the state space.

Remark 2. After we distributed the preprint of this paper, Getzler pointed out to us that in the new version of preprint [7], Getzler and Jones gave a solution of Conjecture 1. They use Fox-Neuwirth's cellular decomposition of the configuration spaces to construct a cellular model $\mathcal{E}_{\in}$ for the operad of configuration spaces, which is homotopically equivalent to the little squares operad, and show that the dot product (1) and the braces (3) determine on $\mathcal{C}^{\bullet}(A)$ the structure of an algebra over the DG operad $\mathcal{E}_{\in}$.

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