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A FORMULA FOR MUMFORD MEASURE IN SUPERSTRING THEORY

A. A. Voronov UDC 517.43+514.84

Polyakov quantization of bosonic structure leads to integration, over the moduli space of algebraic curves, of a measure which is initially expressed in terms of determinants of Laplace operators. After Manin [2] had observed numerical coincidences in string theory and curve moduli theory, Belavin and Knizhnik [3] expressed Polyakov measure as the modulus squared of a holomorphic measure, now known as the Mumford form. This enabled Manin [4] and Beilinson and Manin [5] to express Polyakov measure in terms of the values of differentials at points of curves. In this note we propose a formula for the analog of the Mumford form in superstring theory. In the first part we prove a conjecture of Manin [2, 5]: $\lambda_{3/2}=\lambda_{1/2}^5$. As in the bosonic case [5], this proof will be used in the second part of the note to derive the fundamental formula. We will work with the definition of SUSY-curves (superconformal mappings) due to Baranov and Shvarts [1]. Polyakov supermeasure is defined in [1], but the super-version of the Belavin–Knizhnik theorem is as yet unknown and it is not clear how to do summation of Mumford superforms over spinor structures.

1. Berezinians. Let $\pi: X \to S$ be a smooth proper morphism of complex supervarieties. Then, as in the even case, for any coherent sheaf $\mathcal{H}$ on X which is flat over S, one can define on S a sheaf $B(\mathcal{H})$ of rank 1|0 or 0|1, with the following properties:

1) If all $R^i\pi_*\mathcal{H}$ are locally free, then $B(\mathcal{H}) = \otimes\,(\mathrm{Ber}\,R^i\pi_*\mathcal{H})^{(-1)^i}$;

2) If the sequence $0\to\mathcal{H}'\to\mathcal{H}\to\mathcal{H}''\to 0$ is exact, then $B(\mathcal{H}) = B(\mathcal{H}')\otimes B(\mathcal{H}'')$ (throughout this note, equality means canonical isomorphism).

2. Deligne's Isomorphism in the Supercase. Proposition. Let π be the morphism of Sec. 1, of relative dimension 1|1. Then for any invertible sheaves $\mathcal{L}$ and $\mathcal{M}$ on X (i.e., sheaves of sections of bundles of rank 1|0), $B(\mathcal{L}\otimes\mathcal{M}) = B(\mathcal{O}_X)^{-1}\otimes B(\mathcal{L})\otimes B(\mathcal{M})$.

The proof will be carried out for the most important case, in which $\mathcal{M}$ has a global holomorphic section t defining an effective relative Cartier divisor D on $X \to S$.

The sequence of sheaves $0\to\mathcal{O}_X\xrightarrow{t}\mathcal{M}\to\mathcal{M}|_D\to 0$ is exact, hence $B(\mathcal{M}) = B(\mathcal{O})\otimes B(\mathcal{M}|_D)$. Obviously, $B(\mathcal{M}|_D) = \mathrm{Ber}_{\mathcal{O}_S}(\mathcal{M}|_D)$. Similarly, $0\to\mathcal{L}\xrightarrow{t}\mathcal{L}\otimes\mathcal{M}\to\mathcal{L}\otimes\mathcal{M}|_D\to 0$ implies $B(\mathcal{L}\otimes\mathcal{M}) = B(\mathcal{L})\otimes \mathrm{Ber}(\mathcal{L}\otimes\mathcal{M}|_D)$. In addition, for any invertible sheaves $\mathcal{K}$ and $\mathcal{N}$ on X, $\mathrm{Ber}(\mathcal{K}|_D) = \mathrm{Ber}(\mathcal{N}|_D)$ (this isomorphism is locally defined on S by multiplication on Ber s, where s is any section of the sheaf $\mathcal{N}\otimes\mathcal{K}^*$ different from 0 and ∞ on D_{red}). Applying this assertion to the pair $\mathcal{M}, \mathcal{L}\otimes\mathcal{M}$, we obtain $B(\mathcal{M})\otimes B(\mathcal{O})^{-1} = B(\mathcal{L}\otimes\mathcal{M})\otimes B(\mathcal{L})^{-1}$.

3. Mumford's Formula in the Supercase. Under the assumptions of Sec. 2, suppose there exists a dualizing sheaf ω, i.e., a sheaf of rank 0|1 equipped with the trace morphism tr: $R^1\pi_*\omega\to\mathcal{O}_S$ and defining a Sérre duality, i.e., a nondegenerate pairing $R^i\pi_*\mathcal{F}\otimes R^{1-i}\pi_*(\mathcal{F}^*\otimes\omega)\to R^1\pi_*\omega\xrightarrow{\mathrm{tr}}\mathcal{O}_S$ for locally free $\mathcal{F}$. Let $\lambda_{i/2} = B(\omega^i)$. Clearly, $\lambda_{1/2} = \lambda_0$.

THEOREM. $\lambda_{i/2} = \lambda_{1/2}^{(-1)^{i-1}(2i-1)}$.

Proof. Applying Proposition 2 to $\mathcal{L} = \Pi^i\omega^i$, $\mathcal{M} = \Pi\omega$, (Π is a parity change), we obtain $\lambda_{(i+1)/2}^{(-1)^{i+1}} = \lambda_0^{-1}\otimes\lambda_{i/2}^{(-1)^i}\otimes\lambda_{1/2}^{-1}$. Induction on i, using $\lambda_{1/2}=\lambda_0$, completes the proof.

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4. Application to SUSY-Families. Let $\pi\colon X\to S$ be an SUSY-family, i.e., a smooth proper morphism of relative dimension $1|1$ together with a distribution $\mathcal{T}_1\subset\mathcal{T}X/S$ of rank $0|1$ such that the morphism $[,]\bmod\mathcal{T}_1\colon \mathcal{T}_1\otimes\mathcal{T}_1\to\mathcal{T}/\mathcal{T}_1$ is an isomorphism. In this case the sheaf of holomorphic semiforms $\omega\colon=\mathcal{T}_1^*\cong\operatorname{Ber}X/S$ is a dualizing sheaf (see [6]). By Theorem 3, on any SUSY-family we have $\lambda_{3/2}=\lambda_{1/2}^5$.

5. Formula for Mumford Supermeasure. Mumford supermeasure μ is defined as the image of $1\in\mathcal{O}_S$ under the isomorphism $\mathcal{O}_S=\lambda_{3/2}\otimes\lambda_{1/2}^{-5}$.

Our problem is to write down an expression for μ in terms of given local bases of the sheaves $R^i\pi_*\omega^j$, $i=0,1$, $j=1,2,3$. We shall assume that all the $R^i\pi_*\omega^j$ are locally free, since the notion of basis is defined only in the case that $(\Pi\omega)_{\rm red}$ is an odd nondegenerate θ-characteristic and the genus g of the curve $X_{\rm red}$ is greater than 1. Without these assumptions the formula is found by the same methods but is more cumbersome.

1) Bases. Let ν be an odd global section of the sheaf ω such that $\nu_{\rm red}$ is a nonzero section of $\omega_{\rm red}$ with zero divisor $\{P_1,\ldots,P_{g-1}\}$; let $D=\{\nu=0\}$. Pick coordinates (z_j,ζ_j) in the neighborhood of P_j so that the form $dz_j-\zeta_j d\zeta_j$ vanishes on $\mathcal{T}_1$ and $\nu=(z_j\zeta_j+O(z_j\zeta_j))d\zeta_j$.

Bases in $\operatorname{Ber}(\omega^j|_D)$, $j=1,2,3$. Let $\{\delta_i^k d\zeta_k^j\mid\delta_i^k\zeta_k d\zeta_k^j\}_{i=1}^{g-1}$ be a basis of $\omega^j|_D$. The basis in $\operatorname{Ber}(\omega^j|_D)$ is the element $\delta_{j/2}=\operatorname{Ber}(\delta_i^k d\zeta_k^j\mid\delta_i^k\zeta_k d\zeta_k^j)$.

Basis in λ_0. Let $\{1\mid\xi\}$ be a basis of $\pi_*\mathcal{O}_X$ and $\{\varphi_1,\ldots,\varphi_{g-1},\nu\xi\mid\nu\}$ a basis of $(R^1\pi_*\mathcal{O}_X)^*=\pi_*\omega$ Let $\xi=\xi_j z_j\zeta_j+O(z_j^2)$, $\varphi_i=(\varphi_{ij}^1+\varphi_{ij}^0\zeta_j+O(z_j))d\zeta_j$, where $\varphi_{ij}^0\in\mathcal{O}_{S,0}$, $\varphi_{ij}^1\in\mathcal{O}_{S,1}$, $i,j=1,\ldots,g-1$ and $\varphi_{0,1}^{ij}$: $\Sigma\varphi_0^{ij}\varphi_{kj}^0+\varphi_1^{ij}\varphi_{kj}^1=\delta_k^i$. The basis in λ_0 is defined to be $d_0=\operatorname{Ber}(1\mid\xi)\otimes\operatorname{Ber}(\varphi_1,\ldots,\varphi_{g-1},\nu\xi\mid\nu)$.

Basis in $\lambda_{1/2}$. We choose the Sérre-dual basis $d_{1/2}=d_0$.

Basis in λ_1. Let $\{\nu^2,\chi_1,\ldots,\chi_{g-1}\mid\nu\varphi_1,\ldots,\nu\varphi_{g-1},\nu^2\xi,\psi_1,\ldots,\psi_{g-2}\}$ be a basis of $\pi_*\omega^2$, $\{\xi/\nu\}$ a basis of $(R^1\pi_*\omega^2)^*=\pi_*\omega^{-1}$. Let $\chi_i=(\chi_{ij}^0+\chi_{ij}^1\zeta_j+O(z_j))d\zeta_j^2$, $i,j=1,\ldots,g-1$, $\psi_i=(\psi_{ij}^1+\psi_{ij}^0\zeta_j+O(z_j))d\zeta_j^2$, $i=1,\ldots,g-2$, $j=1,\ldots,g-1$. The basis in λ_1 is the element $d_1=\operatorname{Ber}(\nu^2,\chi_1,\ldots,\chi_{g-1}\mid\nu\varphi_1,\ldots,\nu\varphi_{g-1},\nu^2\xi,\psi_1,\ldots,\psi_{g-2})\otimes\operatorname{Ber}(\xi/\nu)$.

Basis in $\lambda_{3/2}$. Observing that $R^1\pi_*\omega^3=0$, we construct a basis in $\lambda_{3/2}$ with the help of a basis of $\pi_*\omega^3$: $d_{3/2}=\operatorname{Ber}(\nu^2\varphi_1,\ldots,\nu^2\varphi_{g-1},\nu^3\xi,\nu\psi_1,\ldots,\nu\psi_{g-2},\sigma_1,\ldots,\sigma_{g-1}\mid\nu^3,\nu\chi_1,\ldots,\nu\chi_{g-1},\rho_1,\ldots,\rho_{g-2})$, where

$$\rho_i=(\rho_{ij}^0+\rho_{ij}^1\zeta_j+O(z_j))d\zeta_j^3,\ i=1,\ldots,g-2,\ j=1,\ldots,g-1,$$
$$\sigma_i=(\sigma_{ij}^1+\sigma_{ij}^0\zeta_j+O(z_j))d\zeta_j^3,\ i,j=1,\ldots,g-1.$$

2) Relations between $d_{j/2}$ and $\delta_{j/2}$. Identifying $\lambda_{1/2}^{-1}=\lambda_0\otimes\operatorname{Ber}^{-1}(\omega|_D)$, $\lambda_1=\lambda_{1/2}^{-1}\otimes\operatorname{Ber}(\omega^2|_D)$, $\lambda_{3/2}^{-1}=\lambda_1\otimes\operatorname{Ber}^{-1}(\omega^3|_D)$, respectively, we have $d_{1/2}^{-1}=d_0\operatorname{Ber}_{1/2}\delta_{1/2}^{-1}$, $d_1=d_{1/2}^{-1}\operatorname{Ber}_1\delta_1$, $d_{3/2}^{-1}=d_1\operatorname{Ber}_{3/2}\delta_{3/2}^{-1}$. Here

$$\operatorname{Ber}_{1/2}=\operatorname{Ber}\left(\begin{array}{c|c}\varphi_0^{ij}&\varphi_1^{ij}\\\hline\varphi_{ij}^1&\varphi_{ij}^0\end{array}\right),$$

$$\operatorname{Ber}_1=\operatorname{Ber}\left(\begin{array}{c|c}\chi_{ij}^0&\chi_{ij}^1\\\hline\psi_{ij}^1&\psi_{ij}^0\\0\ \ldots\ 0&0\ \ldots\ 0\ \ 1\end{array}\right),\quad \operatorname{Ber}_{3/2}=\operatorname{Ber}\left(\begin{array}{c|c}\rho_{ij}^0&\rho_{ij}^1\\\xi_1^{-1}0\ \ldots\ 0&0\ \ldots\ 0\\\hline\sigma_{ij}^1&\sigma_{ij}^0\end{array}\right).$$

Finally, under the isomorphism $\operatorname{Ber}^{(-1)^{j-1}}(\omega^j|_D)=\operatorname{Ber}(\omega|_D)$ $\delta_{j/2}^{(-1)^{j-1}}=\delta_{1/2}$, $j=2,3$.

3) Computation of μ. By 2), $d_{3/2}=d_{1/2}^5\operatorname{Ber}_{3/2}^{-1}\operatorname{Ber}_1^{-1}\operatorname{Ber}_{1/2}^2$, whence $\mu=d_{3/2}d_{1/2}^{-5}\operatorname{Ber}_{3/2}\operatorname{Ber}_1\operatorname{Ber}_{1/2}^{-2}$.

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