Notation:

- \( \mathbb{N} \) is the set of positive integers;
- \( \mathbb{R} \) is the set of all real numbers;
- \( \mathbb{Q} \) is the set of all rational numbers.

1. Let \( A \) be a set of real numbers.
   (a) (10 points) Express the statement
   
   \( \text{If there is a rational number in } A \text{ then there is a largest number in } A \)

   in abbreviated form, using some of the abbreviations \( \neg, \implies, \& \), \( \lor \), \( \exists \), and \( \forall \), and other common math symbols such as \( \in, \notin, <, \leq \), and \( \mathbb{Q} \) and \( A \).

   **Solution:** Let:
   
   - \( P \) be the statement: \( \text{there is a rational number in } A \)
   - \( Q \) be the statement: \( \text{there is a largest number in } A \).

   Then the original statement is equivalent to: \( P \implies Q \).

   \( P \) can be written in abbreviated form as: \( \exists x (x \in A \& x \in \mathbb{Q}) \).

   Similarly, \( Q \) can be written in abbreviated form as: \( \exists x \in A \forall y \in A (x \leq y) \).

   So the original statement can be written as:
   
   \[
   \exists x (x \in A \& x \in \mathbb{Q}) \implies \exists x \in A \forall y \in A (x \leq y) .
   \]

   (b) (5 points) Write out in abbreviated form the negation of the statement you wrote in (a). Then simplify this statement by moving the negation symbol ‘inside’ as much as possible.

   **Solution:** Note that the negation of \( P \implies Q \) is equivalent to: \( P \& \neg Q \). For the negation \( \neg Q \), the following are equivalent:

   - \( \neg Q \);
   - \( \neg \exists x \in A \forall y \in A (x \leq y) \);
   - \( \forall x \in A \exists y \in A \neg (x \leq y) \);
\[ \forall x \in A \exists y \in A \left( y < x \right). \]

So \( P \land \neg Q \) is equivalent to:

\[ \exists x \left( x \in A \land y \in Q \right) \land \forall x \exists y \left( y < x \right). \]

2. (15 points) Give a proof by induction to show that for every positive integer \( n \), \( 8^n - 3^n \) is a multiple of 5. (In your proof be sure to state precisely the statement you are proving by induction, the Basis Step, the Induction Step, and the Induction Hypothesis.)

**Solution:**

Let \( P(n) \) be the statement: \( 8^n - 3^n \) is a multiple of 5.

We show by induction that \( P(n) \) is true for every \( n \in \mathbb{N} \).

**Basis Step:** \( n = 1 \). \( P(1) \) is the statement \( 8^1 - 3^1 \) is a multiple of 5 (which is true).

**Induction Step:** Given \( n \in \mathbb{N} \), assume \( P(n) \) is true. (This is the Induction Hypothesis. We show \( P(n+1) \) is true. The following are equivalent:

1. \( P(n+1) \) is true;
2. \( 8^{n+1} - 5^{n+1} \) is a multiple of 5;
3. \( 8 \cdot 8^n - 5 \cdot 5^n \) is a multiple of 5;
4. \( 8 \cdot 8^n - 8 \cdot 5^n + 8 \cdot 5^n - 5 \cdot 5^n \) is a multiple of 5;
5. \( 8 \cdot \left( \frac{8^n - 5^n}{\text{multiple of 5}} \right) + (8 - 5) \cdot 5^n \) is a multiple of 5.

Line (5) is true by the induction hypothesis. Hence, line (1) is also true.

3. (25 points) Let \( x \) and \( y \) be real numbers such that \( x < y \). Prove that there is a rational number \( r \) such that \( x < r < y \). (If you use any important theorems in your proof you should mention what they are.)

**Solution:** see the lecture notes.

4. (15 points) Show that there is no positive rational number whose square is equal to 6.
Solution: We give a proof by contradiction. Suppose there is a rational number $p/q$, where $p$ and $q$ are integers, whose square is equal to 6. Then

1. We can assume both $p$ and $q$ are positive integers with no common factor bigger than 1.
2. $(p/q)^2 = 6$.
3. It follows that $p^2 = 6q^2$.
4. By (3), it follows that $p^2$ is even. Hence $p$ must be even.
5. Hence, for some $k \in \mathbb{N}$, $p = 2k$.
6. From (3) and (5), we have $4k^2 = 6q^2$.
7. From (6), $2k^2 = 3q^2$. Then $q^2$ must be even. This implies $q$ is even.
8. For (4) and (7), 2 is a common factor of both $p$ and $q$, which contradicts (1).

5. (a) (5 points) Let $A$ be a set of real numbers. Complete the following definition:

\[ x \text{ is the least upper bound of } A \text{ if } \ldots \]

**Solution:** $x$ is an upper bound of $A$ and there is no smaller upper bound.
(There are several equivalent ways to say the same thing. This may also be written in abbreviated form using quantifiers.)

(b) (5 points) Guess the least upper bound of the set $A$, where

\[ A = \left\{ \frac{2n + 1}{n + 2} : n \in \mathbb{N} \right\} \]

**Solution:** Guess the least upper bound is 2.

(c) (20 points) Using the definition of least upper bound that you gave in part (a), show that your guess is correct.

**Solution:** To show 2 is an upper bound, we have for all $n$,

\[ \frac{2n + 1}{n + 2} < 2 \]
(since \(2n + 1 < 2n + 4\)) and hence every member of \(A\) is less than 2. Next we show that there is no smaller upper bound. Suppose \(x < 2\) we show \(x\) is not an upper bound by finding a member of \(A\) that is bigger than \(x\). That is, we show there is an \(n\) such that
\[
x < \frac{2n + 1}{n + 2}.
\]
Let \(x = 2 - \epsilon\), where \(\epsilon > 0\). The following are equivalent:

1. \(x < \frac{2n+1}{n+2}\);
2. \(2 - \epsilon < \frac{2n+1}{n+2}\);
3. \((2n + 4) - \epsilon \cdot (n + 2) < 2n + 1\);
4. \((2n + 4) - (2n + 1) < \epsilon \cdot (n + 2)\);
5. \(3 < \epsilon \cdot (n + 2)\);
6. \(\frac{1}{n+2} < \frac{\epsilon}{3}\)

So it suffices to chose \(n\) so that \(\frac{1}{n+2} < \frac{\epsilon}{3}\).