Canonical Mackey functors

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Outline

The basics of Mackey functors

Stratification

New theorem
What is a Mackey functor?

A Mackey functor $M$ for a group $G$ over a ring $R$ returns an $R$-module $M(K)$ for each subgroup $K$ of $G$. It has restriction, induction and conjugation maps. Conjugation maps $c_g : M(K) \to M(K)$ are supposed to act trivially if $g \in K$, so $M(K)$ is an $RN_G(K)/K$-module.

Examples

- If $V$ is any $RG$-module the fixed point and fixed quotient functors $FP_V$ and $FQ_V$ are defined by $FP_V(K) := V^K$ and $FQ_V(K) := V_K$.
- For fixed $n$ and $V$, the cohomology groups $M(K) = H^n(K, V)$ define a Mackey functor. Other examples include various Grothendieck groups of group rings, such as $M(K) = \text{Irr}(K)$. 
A Mackey functor over $R$ is a mapping $M : \{\text{subgroups of } G\} \rightarrow R\text{-mod}$ with morphisms $I^H_K : M(K) \rightarrow M(H)$, $R^H_K : M(H) \rightarrow M(K)$, $c_g : M(H) \rightarrow M(\overline{g}H)$ whenever $K \leq H$ and $g \in G$, such that

- $I^H_H, R^H_H, c_h : M(H) \rightarrow M(H)$ are the identity morphisms for all subgroups $H$ and $h \in H$,
- $R^K_J R^H_K = R^H_J$
- $I^H_K I^K_J = I^H_J$ for all subgroups $J \leq K \leq H$,
- $c_g c_h = c_{gh}$ for all $g, h \in G$,
- $R^g_K c_g = c_g R^K_H$
- $I^{gH}_K c_g = c_g I^H_K$ for all subgroups $K \leq H$ and $g \in G$,
- $R^H_J I^K_J = \sum_{x \in [J \setminus H/K]} I^J_{J \cap x K} c_x R^K_{J \cap x K}$ for all subgroups $J, K \leq H$. 
The Mackey functors for $G$ are the objects of an abelian category $\text{Mack}_R(G)$.

These functors are modules for an algebra $\mu_R(G)$ called the Mackey algebra.

There are projective and injective Mackey functors, simple Mackey functors, blocks of Mackey functors, etc.

Induction and restriction of Mackey functors between groups $H \leq G$ can be defined using the morphism $\mu_R(H) \to \mu_R(G)$. The are both the left and right adoint of each other, so are exact and preserve projectives and injectives.

Simple Mackey functors

**Theorem (Thévenaz-Webb (1990))**

Let $H \leq G$ and let $V$ be a simple $R[N_G(H)/H]$-module. Then $(\text{Inf}_{N(H)/H}^{N(H)} FP_V)^{\uparrow G}_{N(H)}$ has a unique simple subfunctor $S_{H,V}$. These functors $S_{H,V}$ form a complete list of the simple Mackey functors.

The set of pairs $H, V$ that index the simple Mackey functors comes with a natural preorder given by inclusion of the subgroups $H$. 


Did you ever see a monkey factor?
The $\Delta$ and $\nabla$ functors: 2001 theory

For any subgroup $H \leq G$ and $RN_G(H)/H$-module $U$ we define

$\Delta_{H,U} = (\inf_{N(H)/H}^{N(H)} FQ_U) \uparrow_G^{N(H)}$ and

$\nabla_{H,U} = (\inf_{N(H)/H}^{N(H)} FP_U) \uparrow_G^{N(H)}$

Proposition

There are formulas:

$$\Delta_{H,U}(K) = \bigoplus_{g \in [K \backslash N_G(H,K)/N_G(H)]} U_{N_{Kg}(H)},$$

and

$$\nabla_{H,U}(K) = \bigoplus_{g \in [K \backslash N_G(H,K)/N_G(H)]} U^{N_{Kg}(H)}.$$

Thus $\Delta_{H,U}(H) = U = \nabla_{H,U}(H);$ both $\Delta_{H,U}(K)$ and $\nabla_{H,U}(K)$ vanish unless $K$ contains a conjugate of $H$. 
Adjoint characterizations

Proposition

\( U \mapsto \nabla_{H,U} \) is right adjoint to taking the **Brauer quotient**
\( M \mapsto \overline{M}(H) \).

Similarly \( U \mapsto \Delta_{H,U} \) is left adjoint to taking the **restriction kernel**
\( \underline{M}(H) \).
The categories $\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$

We will assume from now on that $R$ is a complete $p$-local ring and $U$ is a $p$-permutation $RN_G(H)/H$-module.

Let $\mathcal{F}(\Delta)$ be the full subcategory of Mackey functors that have a finite filtration with factors $\Delta_{H, U}$ where $H \leq G$ and $U$ is a $p$-permutation $RN_G(H)/H$-module.

Similarly, $\mathcal{F}(\nabla)$ is the category with $\nabla$ factors.
Ext vanishing of $\Delta$ and $\nabla$

Let $R$ be a complete $p$-local ring. Let $H$ and $K$ be subgroups of $G$ and let $U$ and $W$ be $p$-permutation modules for $R[N_G(H)/H]$ and $R[N_G(K)/K]$.

Theorem

- $\operatorname{Ext}^1_{\mu_R(G)}(\Delta_{H,U}, \Delta_{K,W}) = 0$ unless $H >_G K$.
- $\operatorname{Ext}^1_{\mu_R(G)}(\Delta_{H,U}, \nabla_{K,W}) = 0$ unless $H =_G K$.
- $\operatorname{Ext}^1_{\mu_R(G)}(\nabla_{H,U}, \nabla_{K,W}) = 0$ unless $H <_G K$.

Theorem

- $\operatorname{Hom}_{\mu_R(G)}(\Delta_{H,U}, \Delta_{K,W}) = 0$ unless $H \geq_G K$.
- $\operatorname{Hom}_{\mu_R(G)}(\Delta_{H,U}, \nabla_{K,W}) = 0$ unless $H =_G K$.
- $\operatorname{Hom}_{\mu_R(G)}(\nabla_{H,U}, \nabla_{K,W}) = 0$ unless $H \leq_G K$. 
Corollary

$\mathcal{F}(\Delta)$ and $\mathcal{F}(\nabla)$ are functorially finite (Ringel). They are closed under direct summands, hence have relative almost split sequences.
Mackey functors in $\mathcal{F}(\Delta)$ for $C_4$ over $\mathbb{F}_2$

<table>
<thead>
<tr>
<th>$(H,U)$</th>
<th>$1, R$</th>
<th>$1, \frac{R}{R}$</th>
<th>$1, RC_4$</th>
<th>$C_2, R$</th>
<th>$C_2, \frac{R}{R}$</th>
<th>$C_4, R$</th>
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<tbody>
<tr>
<td>Mackey functors $\Delta_{H,U}$</td>
<td>$\frac{1}{2}$</td>
<td>$\frac{1}{4}$</td>
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<td>Ext-injectives $T_{H,U}$</td>
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<tr>
<td>Ext-projectives $\Pi_{H,U}$</td>
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Relative AR quiver of $\mathcal{F}(\Delta)$ for $C_4$ over $\mathbb{F}_2$
Theorem

Finitely generated projective Mackey functors lie in $\mathcal{F}(\Delta)$. Finitely generated injective Mackey functors lie in $\mathcal{F}(\nabla)$.

Theorem

The indecomposable Ext-injective and Ext-projective objects in $\mathcal{F}(\Delta)$ are precisely the Ext-injective hulls $I^\Delta_{H,U}$ and Ext-projective covers $\nabla^\Delta_{H,U}$ of the $\Delta_{H,U}$. In any $\Delta$-filtration, $I^\Delta_{H,U}$ always has $\Delta_{H,U}$ at the bottom and $\nabla^\Delta_{H,U}$ always has $\Delta_{H,U}$ at the top.
Reformulation of Alperin’s weight conjecture

Theorem

- $\Pi_{H,U} = \Delta_{H,U} = I_{H,U} \iff H = 1$ and $U$ is projective. 
  *(In this case, $\Pi_{H,U}$ is projective.)*
- $S_{H,U} = \Delta_{H,U} = I_{H,U} \iff (H, U)$ is a weight.

Corollary

Alperin’s weight conjecture holds for $G$ if and only if, among the Mackey functors $\Delta_{H,U} = I_{H,U}$, the number that are projective equals the number that are simple.
New theorem

Theorem (2016)

- \( \mathcal{F}(\Delta) \perp \subseteq \mathcal{F}(\nabla) \) and \( \perp \mathcal{F}(\nabla) \subseteq \mathcal{F}(\Delta) \).
- \( \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) = \mathcal{F}(\Delta) \cap \mathcal{F}(\Delta) \perp = \mathcal{F}(\nabla) \cap \perp \mathcal{F}(\nabla) \).
- The indecomposable objects in \( \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \) are precisely the \( \text{Ext} \)-injective hulls \( I^\Delta_{H,V} \) of the \( \Delta_{H,V} \) in \( \mathcal{F}(\Delta) \), and they are precisely the \( \text{Ext} \)-projective covers \( \Pi^\nabla_{H,V} \) of the \( \nabla_{H,V} \) in \( \mathcal{F}(\nabla) \).
- \( I^\Delta_{H,V} \cong \Pi^\nabla_{H,V} \).
- \( \mathcal{F}(\Delta) \cap \mathcal{F}(\nabla) \) is self-dual: \( (I^\Delta_{H,V})^* \cong \Pi^\nabla_{H,V}^* \cong I^\Delta_{H,V}^* \).
Corollary

The $\text{Ext}$-injective hull $I_{H,V}^\Delta$ of $\Delta_{H,V}$ in $\mathcal{F}(\Delta)$ also has a $\nabla$-filtration. In any $\Delta$-filtration of $I_{H,V}^\Delta$ the bottom term must always be $\Delta_{H,V}$. In any $\nabla$-filtration of $I_{H,V}^\Delta$ the top term must always be $\nabla_{H,V}$.

Corollary

Induction and restriction of Mackey functors both preserve $\mathcal{F}(\Delta) \cap \mathcal{F}(\nabla)$.

Corollary

If all the $p$-permutation modules $V$ are self-dual then the Ringel dual algebra $\text{End}(\bigoplus I_{H,V}^\Delta)$ has symmetric Cartan matrix.