The Coinvariant Algebra in Positive Characteristic

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Reflection Groups

Let V be a vector space over k. A linear map $g : V \to V$ is a *reflection* if and only if it has finite order and fixes pointwise a hyperplane. A finite group G is a *reflection* group if and only if G is a group of linear automorphisms of a vector space and is generated by reflections.

In characteristic zero, real reflection groups are Coxeter groups. Over the complex numbers they are given by the list of Shephard and Todd.

In positive characteristic reflections need not be diagonalizable. Transvections such as

$$\begin{pmatrix} 1 & & 0 & * \\ & \ddots & & * \\ 0 & & 1 & * \\ 0 & & & 1 \end{pmatrix}$$

are reflections. Any group generated by transvections is a reflection group, for example $SL(V)$ and $U(V)$ = upper uni-triangular matrices. Also $GL(V)$ is a reflection group since

$$\begin{pmatrix} \lambda & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

is a reflection. The permutation module for Σ_n is exhibits Σ_n as a reflection group.

Invariants

Let $S = S(V^*)$ be the symmetric algebra on V^*, so choosing a basis $x_1, \dots, x_n$ for V^* we identify S as the polynomial ring $k[x_1, \dots, x_n]$. G acts on S by linear substitutions with invariants S^G.

THEOREM.

(1) *(Shephard-Todd, Chevalley) If* $\operatorname{char} k = 0$ *then* V *is a reflection group if and only if* S^G *is a polynomial ring.*

(2) *(Serre) Over any field, if* S^G *is a polynomial ring then* G *is a reflection group.*

The converse of (2) is false and there is a standard example with the Weyl group of type F_4, see the book of Neusel and Smith.

We will assume always that S^G is polynomial, so for example over $\mathbb{F}_{p^n}$,G could be $GL(V)$, a parabolic subgroup of $GL(V)$, $U(V)$, Σ_n acting on the permutation module for $\{1, \dots, n\}$, etc.

The coinvariant algebra

This is $S \otimes_{S^G} k$. Here we regard S as an S^G-module. Consider the ideal $I \lhd S^G$ consisting of invariants of positive degree, so $S^G/I \cong k$. Now $S \otimes_{S^G} k \cong S/S \cdot I$ where $S \cdot I$ is the ideal in S generated by elements of S^G of positive degree. The notation $S_G = S \otimes_{S^G} k$ is sometimes used. There is an action of G on $S \otimes_{S^G} k$.

Example: Σ_3 permutes x, y, z with invariants generated as a polynomial ring by the elementary symmetric polynomials in degrees 1, 2 and 3, namely $e_1 = x + y + z$, $e_2 = xy + xz + yz$, $e_3 = xyz$.

If W is a Weyl group associated to a complex reductive algebra group G with Borel subgroup B, then $H^*(G/B) \cong S \otimes S^W \mathbb{C}$.

THEOREM (Chevalley (char 0), Mitchell (char p)). *If S^G is polynomial then $S \otimes_{S^G} k$ has the same composition factors as kG.*

The two modules are isomorphic if and only if $|G|$ is invertible in k. If S^G is polynomial then S is free as a S^G-module (since S is Cohen-Macaulay, so is free over any polynomial subring of the same dimension). $S = S^G a_1 \oplus \cdots \oplus S^G a_r$ if and only if the images of $a_1, \ldots, a_r$ are a basis for $S/S \cdot I = S \otimes_{S^G} k$.

We have $\mathrm{Hilb}(S, t) = \mathrm{Hilb}(S^G, t) \cdot \mathrm{Hilb}(S \otimes_{S^G} k, t)$ so that

$$\begin{aligned}
\mathrm{Hilb}(S \otimes_{S^G} k, t) &= \frac{\mathrm{Hilb}(S, t)}{\mathrm{Hilb}(S^G, t)} \\
&= \frac{1}{(1-t)^n} \Big/ \frac{1}{(1-t^{d_1}) \cdots (1-t^{d_n})} \\
&= (1 + t + \cdots + t^{d_1 - 1}) \cdots (1 + t + \cdots + t^{d_n - 1})
\end{aligned}$$

and we see that $\dim S \otimes_{S^G} k = d_1 \cdots d_n$. In the case of a Coxeter group this polynomial is also known to be the Poincaré polynomial of the group.

Example. Σ_4 acts on $\{w, x, y, z\}$. $\mathrm{Hilb}(S \otimes_{S^G} k, t) = (1+t)(1+t+t^2)(1+t+t^2+t^3)$.

Regular elements

Let $\bar{k}$ be the algebraic closure of k. A vector $v \in \overline{V} = V \otimes_k \bar{k}$ is *regular* if and only if it is not in any reflecting hyperplane of any reflection in G, if and only if v is fixed by no reflection in G. An element $c \in G$ is *regular* if and only if it has an eigenvector which is regular.

Examples: (Springer, Invent. Math 25 (1974)) With Σ_n permuting $x_1, \ldots, x_n$ in characteristic zero the regular elements are the n-cycles and the $(n-1)$-cycles. (An eigenvector of any shorter cycle is fixed by an involution on two remaining points. An eigenvector of a permutation is an eigenvector of one of its cycles.)

LEMMA (Reiner-Stanton-Webb). *If S^G is polynomial,* $\operatorname{char} k = p$ *and c is regular, then c is p-regular (i.e. has order prime to p).*

Application: the Σ_4 example in characteristics 2 and 3.

PROPOSITION (Reiner-Stanton-Webb). *In $GL_n(\mathbb{F}_q)$ the regular elements are exactly the powers of Singer cycles.*

We define a new action of a regular element $c \in G$ on $\overline{S} := \bar{k} \otimes_k S$. Let $cv = \zeta v$ where $\zeta \in \bar{k}^\times$ and v is regular. The order of c is divisible by the order of ζ (and in fact these orders are equal when S^G is polynomial). Let c act on $\overline{S}_n$ as multiplication by ζ^n. Thus c acts on $\overline{S} \otimes_{\overline{S}^G} \bar{k}$. The action commutes with that of G, and these become $\bar{k}[G \times \langle c \rangle]$-modules. We regard $\bar{k}G$ also as a $\bar{k}[G \times \langle c \rangle]$-module, where G acts by left multiplication and c acts by right multiplication (by c^{-1}).

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THEOREM (Springer (char 0), Reiner-Stanton-Webb (char p)). *As $\bar{k}[G \times \langle c \rangle]$-modules, $\overline{S} \otimes_{S^G} \bar{k}$ and $\bar{k}G$ have the same composition factors.*

Proof. This is a Brauer character calculation. $\square$

Application: $\bar{k}C$ is semisimple and $\bar{k}C = \bar{k}Ce_0 \oplus \cdots \oplus \bar{k}Ce_{d-1}$ where $e_i^2 = e_i$ and for any right $\bar{k}C$-module M, Me_i is the ζ^i eigenspace of c. Thus $\bar{k}G = \bar{k}Ge_0 \oplus \cdots \oplus \bar{k}Ge_{d-1}$ is a direct sum of projective $\bar{k}G$-modules. Since c acts freely on G, these summands all have the same dimension. On $\overline{S} \otimes_{\overline{S}^G} \bar{k}$, the ζ^i-eigenspace is the sum of the homogeneous terms in degrees $r \equiv i \,(\mathrm{mod}\, d)$. This sum has the same $\bar{k}G$-module composition factors as $\bar{k}Ge_i$.

THEOREM. *$\bigoplus_{r \equiv i \,(\mathrm{mod}\, d)} (\overline{S} \otimes_{\overline{S}^G} \bar{k})_r$ has the composition factors of a projective $\bar{k}G$-module.*

Question: Does there exist a filtration of $\bar{k}G$ by $\bar{k}[G \times C]$-modules so that the factors, taken in ascending order, are isomorphic as $\bar{k}[G \times C]$-modules to the homogeneous terms of $\overline{S} \otimes_{\overline{S}^G} \bar{k}$ taken in ascending order?