0.7.1 “Modulus of $z$,” “absolute value of $z$,” and $|z|$ are synonyms. “Real part of $z$” is the same as $\text{Re } z = a$. “Imaginary part of $z$” is the same as $\text{Im } z = b$. The “complex conjugate of $z$” is the same as $\bar{z}$.

0.7.2

0.7.3 a. The absolute value of $2 + 4i$ is $|2 + 4i| = 2\sqrt{5}$. The argument (polar angle) of $2 + 4i$ is $\arccos 1/\sqrt{5}$, which you could also write as $\arctan 2$.

b. The absolute value of $(3 + 4i)^{-1}$ is $1/5$. The argument (polar angle) is $-\arccos(3/5)$.

c. The absolute value of $(1 + i)^5$ is $4\sqrt{2}$. The argument is $5\pi/4$. (The complex number $1 + i$ has absolute value $\sqrt{2}$ and polar angle $\pi/4$. De Moivre’s formula says how to compute these for $(1 + i)^5$.)

d. The absolute value of $1 + 4i$ is $\sqrt{17}$; the argument is $\arccos 1/\sqrt{17}$.

0.7.4 a. $|3 + 2i| = \sqrt{3^2 + 2^2} = \sqrt{13}; \quad \arctan \frac{2}{3} \approx .58803$.

Remark. The angle is in radians; all angles will be in radians unless explicitly stated otherwise.

b. $|(1 - i)^4| = |1 - i|^4 = (\sqrt{2})^4 = 4; \quad \arg((1 - i)^4) = 4 \arg(1 - i) = 4 \left( -\frac{\pi}{4} \right) = -\pi$.

One could also just observe that $(1 - i)^4 = ((1 - i)^2)^2 = (-i)^2 = -4$.

c. $|2 + i| = \sqrt{5}; \quad \arg(2 + i) = \arctan 1/2 \approx .463648$.

d. $|\sqrt{3} + 4i| = \sqrt{\sqrt{25}} \approx 1.2585; \quad \arg \sqrt{3} + 4i = \frac{1}{7} \left( \arctan \frac{4}{3} + \frac{2k\pi}{7} \right)$.

These numbers are $\approx .132471, 1.03007, 1.92767, 2.82526, 3.72286, 4.62046, 5.51806$.

Remark. In this case, we have to be careful about the argument. A complex number doesn’t have just one 7th root, it has seven of them, all with the same modulus but different arguments, differing by integer multiples of $2\pi/7$.

0.7.5 Parts 1–4 are immediate. For part 5, we find

$$(z_1z_2)z_3 = \left( (x_1x_2 - y_1y_2) + i(y_1x_2 + x_1y_2) \right)(x_3 + iy_3)$$

$$= (x_1x_2x_3 - y_1y_2x_3 - y_1x_2y_3 - x_1y_2y_3)$$

$$+ i(x_1x_2y_3 - y_1y_2y_3 + y_1x_2x_3 + x_1y_2x_3),$$
which is equal to
\[ z_1(z_2 z_3) = (x_1 + iy_1)((x_2 x_3 - y_2 y_3) + i(y_2 x_3 + x_2 y_3)) \]
\[ = (x_1 x_2 x_3 - x_1 y_2 y_3) - y_1 y_2 x_3 + y_1 x_2 y_3 \]
\[ + i(y_1 x_2 x_3 - y_1 y_2 y_3 + x_1 y_2 x_3 + x_1 x_2 y_3). \]

Parts 6 and 7 are immediate. For part 8, multiply out:
\[ (a + ib) \left( \frac{a}{a^2 + b^2} - i \frac{b}{a^2 + b^2} \right) = \frac{a^2}{a^2 + b^2} + \frac{b^2}{a^2 + b^2} + i \left( \frac{-ab}{a^2 + b^2} - \frac{ab}{a^2 + b^2} \right) \]
\[ = 1 + i0 = 1. \]

Part 9 is also a matter of multiplying out:
\[ z_1(z_2 + z_3) = (x_1 + iy_1)((x_2 + iy_2) + (x_3 + iy_3)) \]
\[ = (x_1 + iy_1)((x_2 + x_3) + i(y_2 + y_3)) \]
\[ = x_1 (x_2 + x_3) - y_1 (y_2 + y_3) + i(y_1 (x_2 + x_3) + x_1 (y_2 + y_3)) \]
\[ = x_1 x_2 - y_1 y_2 + i(y_1 x_2 + x_1 y_2) + x_1 x_3 - y_1 y_3 + i(y_1 x_3 + x_1 y_3) \]
\[ = z_1 z_2 + z_1 z_3. \]

0.7.6 a. The quadratic formula gives
\[ x = \frac{-i \pm \sqrt{i^2 - 4}}{2} = \frac{-i \pm \sqrt{-5}}{2} = -\frac{i}{2}(-1 \pm \sqrt{5}). \]

b. The quadratic formula gives
\[ x^2 = \frac{-1 \pm \sqrt{1 - 4}}{2} = \frac{1}{2}(-1 \pm i\sqrt{3}). \]

These aren’t any old complex numbers: they are the non-real cubic roots of 1, and their square roots are the non-real sixth roots of 1:
\[ \frac{1}{2}(\pm 1 \pm i\sqrt{3}). \]

Remark. You didn’t have to “notice” that \((-1 + i\sqrt{3})/2\) is a cubic root of 1, the square root could have been computed in the standard way anyway. Why are the solutions 6th roots of 1? Because
\[ x^6 - 1 = (x^2 - 1)(x^4 + x^2 + 1), \]
so all roots of \(x^4 + x^2 + 1\) will also be roots of \(x^6 - 1\).

0.7.7 a. The equation \(|z - u| + |z - v| = c\) represents an ellipse with foci at \(u\) and \(v\), at least if \(c > |u - v|\). If \(c = |u - v|\) it is the degenerate ellipse consisting of just the segment \([u, v]\), and if \(c < |u - v|\) it is empty, by the triangle inequality, which asserts that if there is a \(z\) satisfying the equality, then
\[ c < |u - v| \leq |u - z| + |z - v| = c. \]

Solution 0.7.7: Remember that the set of points such that the sum of their distances to two points is constant, is an ellipse, with foci at those points.
b. Set \( z = x + iy \); the inequality \(|z| < 1 - \text{Re} z\) becomes
\[
\sqrt{x^2 + y^2} < 1 - x,
\]
corresponding to a region bounded by the curve of equation
\[
\sqrt{x^2 + y^2} = 1 - x.
\]
If we square this equation, we will get the curve of equation
\[
x^2 + y^2 = 1 - 2x + x^2, \quad \text{i.e.,} \quad x = \frac{1}{2}(1 - y^2),
\]
which is a parabola lying on its side. The original inequality corresponds to the inside of the parabola.

0.7.9
a. The quadratic formula gives
\[
x = \frac{-i \pm \sqrt{-1-8}}{2},
\]
so the solutions are \( x = i \) and \( x = -2i \).

b. In this case, the quadratic formula gives
\[
x^2 = \frac{-1 \pm \sqrt{1-8}}{2} = \frac{-1 \pm i\sqrt{7}}{2}.
\]
Each of these numbers has two square roots, which we still need to find.

One way, probably the best, is to use the polar form; this gives
\[
x^2 = r(\cos \theta \pm i \sin \theta),
\]
where
\[
r = \frac{\sqrt{1+7}}{2} = \sqrt{2}, \quad \theta = \pm \arccos \left(-\frac{1}{2\sqrt{2}}\right) \approx 1.2094\ldots \text{ radians}.
\]
Thus the four roots are
\[
\pm \sqrt{2}(\cos \theta/2 + i \sin \theta/2) \quad \text{and} \quad \pm \sqrt{2}(\cos \theta/2 - i \sin \theta/2).
\]

c. Multiplying the first equation through by \((1 + i)\) and the second by \(i\) gives
\[
i(1 + i)x - (2 + i)(1 + i)y = 3(1 + i)
\]
\[
i(1 + i)x - y = 4i,
\]
which gives
\[
-(2 + i)(1 + i)y + y = 3 - i, \quad \text{i.e.,} \quad y = i + \frac{1}{3}.
\]
Substituting this value for \( y \) then gives \( x = \frac{7}{3} - \frac{8}{3}i \).

0.7.10

0.7.11  a. These are the vertical line \( x = 1 \) and the circle centered at the origin of radius 3.
b. Use $Z = X + iY$ as the variable in the codomain. Then

$$(1 + iy)^2 = 1 - y^2 + 2iy = X + iY$$

gives $1 - X = y^2 = Y^2/4$. Thus the image of the line is the curve of equation $X = 1 - Y^2/4$, which is a parabola with horizontal axis.

The image of the circle is another circle, centered at the origin, of radius 9, i.e., the curve of equation $X^2 + Y^2 = 81$.

c. This time use $Z = X + iY$ as the variable in the domain. Then the inverse image of the line $\Re z = 1$ is the curve of equation

$$\Re (X + iY)^2 = X^2 - Y^2 = 1,$$

which is a hyperbola. The inverse image of the curve of equation $|z| = 3$ is the curve of equation $|Z|^2 = |Z|^2 = 3$, i.e., $|Z| = \sqrt{3}$, the circle of radius $\sqrt{3}$ centered at the origin.

0.7.12

0.7.13 a. The cube roots of 1 are

$$1, \cos \frac{2\pi}{3} + i\sin \frac{2\pi}{3} = -\frac{1}{2} + \frac{i\sqrt{3}}{2}, \quad \cos \frac{4\pi}{3} + i\sin \frac{4\pi}{3} = -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

b. The fourth roots of 1 are $1, i, -1, -i$.

c. The sixth roots of 1 are

$$1, -1, \frac{1}{2} + \frac{i\sqrt{3}}{2}, \frac{1}{2} - \frac{i\sqrt{3}}{2}, -\frac{1}{2} + \frac{i\sqrt{3}}{2}, -\frac{1}{2} - \frac{i\sqrt{3}}{2}.$$

0.7.14

0.7.15 a. The fifth roots of 1 are

$$\cos \frac{2\pi k}{5} + i\sin \frac{2\pi k}{5}, \text{ for } k = 0, 1, 2, 3, 4.$$

The point of the question is to find these numbers in some more manageable form. One possible approach is to set $\theta = \frac{2\pi}{5}$, and to observe that $\cos 4\theta = \cos \theta$. If you set $x = \cos \theta$, this leads to the equation

$$2(2x^2 - 1)^2 - 1 = x \quad \text{i.e.,} \quad 8x^4 - 8x^2 - x + 1 = 0.$$

This still isn’t too manageable, until you start asking what other angles satisfy $\cos 4\theta = \cos \theta$. Of course $\theta = 0$ does, meaning that $x = 1$ is one root of our equation. But $\theta = 2\pi/3$ does also, meaning that $-1/2$ is also a root. Thus we can divide:

$$\frac{8x^4 - 8x^2 - x + 1}{(x-1)(2x+1)} = 4x^2 + 2x - 1,$$

and $\cos 2\pi/5$ is the positive root of that quadratic equation, i.e.,

$$\cos \frac{2\pi}{5} = \frac{\sqrt{5} - 1}{4}, \text{ which gives } \sin \frac{2\pi}{5} = \frac{\sqrt{10 + 2\sqrt{5}}}{4}.$$
The fifth roots of 1 are now
\[
1, \frac{\sqrt{5} - 1}{4} \pm i \frac{\sqrt{10 + 2\sqrt{5}}}{4}, \quad \frac{\sqrt{5} + 1}{4} \pm i \frac{\sqrt{10 - 2\sqrt{5}}}{4}.
\]

b. It is straightforward to draw a line segment of length \((\sqrt{5} - 1)/4\): construct a rectangle with sides 1 and 2, so the diagonal has length \(\sqrt{5}\). Then subtract 1 and divide twice by 2, as shown in the figure below.

\begin{center}
\includegraphics[width=0.5\textwidth]{figure.png}
\end{center}

\textbf{Figure for solution 0.7.15.}
So if you set \( \delta = \epsilon \), and \(|H| \leq \delta\), then equation (2) is satisfied.

c. We will show that the limit does not exist. In this case, we find

\[
(A + H - A)^{-1} (A + H)^2 - A^2 = H^{-1}(I^2 + AH + HA + H^2 - I^2) = H^{-1}(AH + HA + H^2) = A + H^{-1}AH + H^2.
\]

If the limit exists, it must be 2\( A \): choose \( H = \epsilon I \) so that \( H^{-1} = \epsilon^{-1} I \); then

\[
A + H^{-1}AH + H^2 = 2A + \epsilon I
\]
is close to 2\( A \).

But if you choose \( H = \epsilon \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix} \), you will find that

\[
H^{-1}AH = \begin{bmatrix} 1/\epsilon & 0 \\ 0 & -1/\epsilon \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -A.
\]

So with this \( H \) we have

\[
A + H^{-1}AH + H^2 = A - A + \epsilon H
\]
which is close to the zero matrix.

**1.5.24**

1.6.1 Let \( B \) be a set contained in a ball of radius \( R \) centered at a point \( a \). Then it is also contained in a ball of radius \( R + |a| \) centered at the origin; thus it is bounded.

1.6.2 First, remember that compact is equivalent to closed and bounded so if \( A \) is not compact then \( A \) is unbounded and/or not closed. If \( A \) is unbounded then the hint is sufficient. If \( A \) is not closed then \( A \) has a limit point \( a \) not in \( A \): i.e., there exists a sequence in \( A \) that converges in \( \mathbb{R}^n \) to a point \( a \notin A \). Use this \( a \) as the \( a \) in the hint.

1.6.3 The polynomial \( p(z) = 1 + x^2 y^2 \) has no roots because 1 plus something positive cannot be 0. This does not contradict the fundamental theorem of algebra because although \( p \) is a polynomial in the real variables \( x \) and \( y \), it is not a polynomial in the complex variable \( z \): it is a polynomial in \( z \) and \( \bar{z} \). It is possible to write \( p(z) = 1 + x^2 y^2 \) in terms of \( z \) and \( \bar{z} \). You can use

\[
x = \frac{z + \bar{z}}{2} \quad \text{and} \quad y = \frac{z - \bar{z}}{2i},
\]

and find

\[
p(z) = 1 + \frac{z^4 - 2|z|^4 + \bar{z}^4}{16}
\]

but you simply cannot get rid of the \( \bar{z} \).

1.6.4 If \(|z| \geq 4\), then

\[
|p(z)| \geq |z|^3 - 4|z|^3 - 3|z| - 3 > |z|^5 - 4|z|^3 - 3|z|^3 - 3|z|^3 = |z|^3(|z|^2 - 10) \geq 6 \cdot 4^3.
\]