Math 8245 **Homework 2** **PJW**
Date due: Wednesday October 10, 2018.

All these questions are to be done using GAP

1. Let $p_1, p_2, p_3, \ldots = 2, 3, 5, \ldots$ be the sequence of primes, and define

$$E_n = p_1 \cdot p_2 \cdots p_n + 1.$$

These are the numbers that appear in Euclid's proof that there are infinitely many primes, and have the property that E_n is not divisible by any of $p_1, \ldots, p_n$. Write a program in GAP that prints out which of the numbers E_n are prime, where $1 \leq n \leq 80$.
[This exercise tests use of loops and the GAP functions for integers. Instead of typing in your program live within a GAP session, you could try creating the program in a separate file in the directory from which you started GAP, and read in the file to GAP using Read("filename"); that way you do not need to type everything again when you make corrections.]

2. Consider the groups:

$$\begin{aligned}
g1 &= \langle (1,2,3,4,5,6,7,8), (2,4)(3,7)(6,8) \rangle \\
g2 &= \langle (1,2,3,4,5,6,7,8), (2,8)(3,7)(4,6) \rangle \\
g3 &= \langle (1,2,3,4,5,6,7,8), (2,6)(4,8) \rangle \\
g4 &= \langle (1,4,6,8,10,12,14,15)(2,3,5,7,9,11,13,16), \\
&\qquad (1,2)(3,15)(4,16)(5,14)(6,13)(7,12)(8,11)(9,10) \rangle \\
g5 &= \langle (1,2,3,4,5,6,7,8), (9,10) \rangle.
\end{aligned}$$

(a) For each group compute
(i) a list of the elements of the group,
(ii) a list of the orders of the elements of the group.
(b) Determine whether any of these groups are isomorphic to one another. In each case where more than one of the groups are isomorphic to each other, identify the group as a known group with a name.
[Computer algorithms that establish an isomorphism between two groups are very poor — they basically run through all possible bijections and see if any of them are group homomorphisms. Instead, to show that certain groups are isomorphic here you should identify the groups in question as groups you already know something about, and use some theory to establish isomorphism.]
(c) In the case of the group $g3$, compute the lattice of subgroups. Provide generators for each subgroup. Draw a picture of the lattice of subgroups, where one subgroup is shown immediately below another if one is a maximal subgroup of the other — in other words, draw the Hasse diagram. Identify all the non-abelian subgroups, the center, the derived subgroup.
[There are commands `ConjugacyClassesSubgroups` and `LatticeSubgroups` that I am sure some of you would be tempted to explore in tackling this problem. I suggest

that it would be at least as easy for you to construct the lattice of subgroups by intelligent direct use of the functions I have already shown you in GAP. If you do insist on finding out about the subgroup functions I have just mentioned, be warned that I ask for a lattice of subgroups, not of conjugacy classes of subgroups, so I want all subgroups in the picture. Also, I ask for a picture, not the kind of output that these built-in functions produce.]

3. Let s be the Sylow 2-subgroup of the group

$$g = \langle (1,2,3,4,5,6,7,8,9,10,11),(3,7,11,8)(4,10,5,6)\rangle.$$

a) Obtain a permutation representation of s on 8 symbols.
b) It is the case that s is isomorphic to one of the groups in question 2. To which one is it isomorphic?
c) Construct a subgroup of g of order divisible by 2, that is not a 2-group and that does not contain a Sylow 2-subgroup of g (any such subgroup will do!).
[This group g is the Mathieu group M_{11}. Use SylowSubgroup, Orbits, Action.]

4. Repeat question 3. with the group

$$\langle (1,2,3,4,5,6,7,8,9,10,11,12,13),(2,3)(5,10)(7,11)(9,12)\rangle.$$

[This group is $PSL(3,3)$.]

5. Show that the group $\langle (1,5)(2,6),(1,3)(4,6),(2,3)(4,5)\rangle$ is isomorphic to S_4.