Math 8300 Homework 1 PJW

Date due: Monday January 30, 2017. We will discuss these questions on Wednesday 2/1/2017

1. (2 pts) Let $M$ be a $kG$-module. Show that $M$ admits a non-singular $G$-invariant bilinear form if and only if $M \cong M^*$ as $kG$-modules.

2. Let $M$ be a $kG$-module and let $B$ be the vector space of bilinear forms $M \times M \to k$. 
   a) (2 pts) For each $g \in G$ we may construct two new bilinear forms $(\cdot, \cdot)_g^1 : v, w \mapsto \langle vg, wg \rangle$, and $(\cdot, \cdot)_g^2 : v, w \mapsto \langle vg^{-1}, wg^{-1} \rangle$. One of these definitions makes $B$ into a $kG$-module via $(\cdot, \cdot)_g^i \cdot g = (\cdot, \cdot)_g^i$, $i = 1$ or 2. Which value of $i$ achieves this?
   
   b) (0 pts) We note without further comment that a bilinear form is $G$-invariant $\iff$ it is fixed in this $G$-action.
   
   c) (2 pts) Taking a standard basis for $M$ and for $B$ we may express a bilinear form $f$ by its Gram matrix $A_f$, and the action of $g \in G$ on $M$ by its matrix $\rho(g)$. Which of the following gives the right action of $G$ on $B$ (pun intended): (i) $A_f \mapsto \rho(g)^T A_f \rho(g)$, or (ii) $A_f \mapsto \rho(g) A_f \rho(g)^T$?

3. Let $G = C_3 = \langle g \rangle$ be cyclic of order 3 and let $k = \mathbb{F}_3$. We define $M_2 = ke_1 \oplus ke_2$ to be a 2-dimensional space acted on by $g$ via the matrix $\rho(g) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$.
   
   a) (1 pts) Find the matrix via which $g$ acts on the space $B$ of bilinear forms $M \times M \to k$.
   
   b) (2 pts) Show that the space of $G$-invariant bilinear forms has dimension 2.
   
   c) (1 pts) Show that $M_2 \cong M_2^*$ as $kG$-modules and find a $G$-invariant non-degenerate form on $M_2$.
   
   d) (2 pts) Show that $M_2$ does not admit any symmetric $G$-invariant non-degenerate bilinear form, but that it does admit a skew-symmetric such form.

4. (1 pts) Let $U$ be a $kG$-submodule of the $kG$-module $M$. Show that $U^\perp$ is a $kG$-submodule of $M^*$.
   
   (3 pts) Suppose further that $M$ comes supplied with a non-singular $G$-invariant bilinear form. Show that $U^\perp \cong U^\circ$ as $kG$-modules. Deduce that the isomorphism type of $U^\perp$ is independent of the choice of non-singular $G$-invariant bilinear form.

5. (2 pts) Let $H$ be a subgroup of a group $G$, and write
   
   $H \backslash G = \{ Hg \mid g \in G \}$

   for the set of right cosets of $H$ in $G$. There is a permutation action of $G$ on this set from the right, namely $(Hg_1)g_2 = Hg_1g_2$. Let $\mathbf{H} = \sum_{h \in H} h \in kG$ denote the sum of the elements of $H$, as an element of the group ring of $G$. Show that the permutation
module $k[H \setminus G]$ is isomorphic as an $kG$-module to the submodule $\mathcal{P} \cdot kG$ of $kG$.

[Facts about permutation modules for those new to representation theory. These comments will not help with the question in any way that I can see.]

a) If $\Omega$ is a transitive $G$-set and $\omega \in \Omega$ with stabilizer $H = \text{Stab}(\omega)$ then $\Omega \cong H \setminus G$ as $G$-sets.

b) $k[H \setminus G] \cong k \uparrow^G_H$ as $kG$-modules.]

6. (3=1+2 pts) Let $V$ be the subspace of the 10-dimensional space $k^{10}$ over the field $k$ which has as a basis the vectors

\[
\begin{align*}
[0, & 1, -1, -1, 1, 0, 0, 0, 0] \\
[1, & 0, -1, -1, 0, 1, 0, 0, 0] \\
[0, & 1, -1, 0, 0, 0, -1, 1, 0, 0] \\
[1, & 0, -1, 0, 0, -1, 0, 1, 0] \\
[1, & 0, 0, 0, -1, 0, -1, 0, 0, 1].
\end{align*}
\]

With respect to this basis of $V$, write down the Gram matrix for the bilinear form on $V$ which is the restriction of the standard bilinear form on $k^{10}$. Supposing further that $k$ has characteristic 3, determine the dimension of the space $V/(V \cap V^\perp)$. [$V$ is the Specht module $S^{[3,2]}$.]

Extra questions for practice with partitions: do not hand in.

7. Find all pairs of partitions of 7 which are not comparable in the dominance ordering, i.e. pairs $(\lambda, \mu)$ for which it is neither true that $\lambda \succeq \mu$ nor $\mu \succeq \lambda$.

8. Determine all natural numbers $n$ and partitions $\lambda$ of $n$ for which the number of $\lambda$-tabloids is 12 or fewer (and hence gain an impression of the examples that it is feasible to work by hand).