These questions can all be done using technology presented in class. It would be possible to do some of them in a different way, perhaps by studying various texts. The point about these questions is that they reinforce what is done in class, and I prefer it if you use the methods I have taught.

1. (4 pts) Show by example that the homomorphism $FGL(E) \to S_F(n,r)$ given by the representation of $GL(E)$ on $E^\otimes r$ need not be surjective if the field $F$ is not infinite.

2. (4 pts) Show by example that it is possible to find a group $G$, a $\mathbb{Z}G$-module $U$ and a prime $p$ so that the ring homomorphism $\text{End}_{\mathbb{Z}G} \to \text{End}_{\mathbb{F}_pG}(U/pU)$ is not surjective.

3. (2 pts) Let $M$ be a module for a ring $A$, and suppose that $M$ has just two composition factors and is indecomposable. Show that $M$ has a unique submodule, other than $0$ and $M$.

4. True or false? Provide either a proof or a counterexample for each part. Let $t$ be a $\lambda$-tableau.
   (a) (2 pts) In any direct sum decomposition of $M^\lambda$ as a direct sum of indecomposable $\mathbb{F}_pS_r$-modules, there is a unique summand on which $\kappa_t$ has non-zero action.
   (b) (2 pts) Furthermore, if $Y^\mu$ is a Young module for $\mathbb{F}_pS_r$ which has a submodule isomorphic to $S^\lambda$ then $\lambda \trianglerighteq \mu$.
   (c) (2 pts) Determine whether or not this gives a proof that the various Young modules $Y^\lambda$, as $\lambda$ ranges through partitions of $r$, are all non-isomorphic.

5. In this question, tableaux may have repeated entries. Let $\lambda$ be a partition of $r$, and let $\mu$ be any sequence of non-negative integers, whose sum is $r$. We say that a $\lambda$-tableau $T$ has type $\mu$ if, for every $i$, the number $i$ occurs $\mu_i$ times in $T$. For example, $T = \begin{array}{cccc} 2 & 2 & 1 & 1 \\ 1 & \end{array}$ is a $[4,1]$-tableau of type $[3,2]$. We will number the positions in $T$ according to some tableau with distinct entries, such as

$$t = \begin{array}{cccc} 1 & 2 & 3 & 4 \\ 5 & \end{array},$$

but it could have been some other such tableau.
   (a) (2 pts) Show that the set of $\lambda$-tableaux of type $\mu$ is in bijection with the set of $\mu$-tabloids.

We now let $S_r$ act on the $\lambda$-tableaux of type $\mu$ by permuting the positions of the entries. Thus if $T = \begin{array}{cccc} 2 & 2 & 1 & 1 \\ 1 & \end{array}$ then $T(1,5) = \begin{array}{cccc} 1 & 2 & 1 & 1 \\ 2 & \end{array}$ and $T(1,5,2) = \begin{array}{cccc} 2 & 1 & 1 & 1 \\ 2 & \end{array}$ since $(1,5,2) = (1,5)(1,2)$. We say that $T_1$ and $T_2$ are row equivalent if $T_2 = T_1 \pi$ for some permutation in the row stabilizer of the $\lambda$-tableau $t$. 

1
(b) (2 pts) Show that the row equivalence classes of \( \lambda \)-tableaux of type \( \mu \) are in bijection with the double cosets \( S_\mu \backslash S_r / S_\lambda \).

(c) (2 pts) Show that for each \( \lambda \)-tableau \( T \) of type \( \mu \) there is a \( RS_r \)-module homomorphism \( \theta_T : M^\lambda \to M^\mu \) such that \( \theta_T(\{t\}) = \sum \{ T_i \mid T_i \text{ is row equivalent to } T \} \). Thus, in the above example,

\[
\theta_T(\{t\}) = \begin{pmatrix}
2 & 2 & 1 & 1 \\
1 & 1 & 2 & 1 & 2 & 1 & 1 & 2
\end{pmatrix}.
\]

(d) (2 pts) Show that, as \( T \) ranges over the row equivalence classes of \( \lambda \)-tableaux of type \( \mu \) the homomorphisms \( \theta_T \) give a basis for \( \text{Hom}_{RS_r}(M^\lambda, M^\mu) \).

6. In this question you may assume that there is a decomposition of the group algebra \( \mathbb{F}_2S_3 \cong Y^{[1^3]} \oplus Y^{[2,1]} \oplus Y^{[2,1]} \) and that \( Y^{[1^3]} \) has dimension 2, and has a unique \( \mathbb{F}_2S_3 \)-submodule of dimension 1. Let \( E = \mathbb{F}_2^3 \) be a 3-dimensional space over \( \mathbb{F}_2 \).

(a) (2 pts) Express \( E \otimes^3 \) as a direct sum of modules \( M^\lambda \), determining the multiplicity of each \( M^\lambda \) summand.

(b) (2 pts) Make a table with rows and columns indexed by the partitions of 3, whose \( \lambda, \mu \) entry is the number of double cosets \( |S_\lambda \backslash S_3 / S_\mu| \).

(c) (2 pts) Compute the dimension of \( S_{\mathbb{F}_2}(3,3) \).

(d) (2 pts) Compute the dimensions of the simple modules for \( S_{\mathbb{F}_2}(3,3) \).

(e) (2 pts) Show that, as \( S_{\mathbb{F}_2}(3,3) \)-modules, the symmetric tensors \( ST^3(E) \) is indecomposable, but that \( E \otimes^3 \) is the direct sum of three indecomposable submodules, and find their dimensions.

Extra question: do not hand in

7. Find a basis for the space of homomorphisms \( \text{Hom}_{FS_5}(M^{(3,2)}, M^{(2,1,1,1)}) \). For each element \( \theta \) in your basis, compute the effect of \( \theta \) on the tabloid

\[
\begin{array}{c}
1 & 2 & 3 \\
4 & 5
\end{array}
\]

8. Let \( U = U_1 \oplus U_2 = U'_1 \oplus U'_2 \) be two direct sum decompositions of a module for an algebra \( A \), and let \( 1_U = f_1 + f_2 = f'_1 + f'_2 \) be the corresponding expressions for \( 1_u \) as sums of orthogonal idempotents in \( \text{End}_A(U) \). Show that (a) \( U_1 \cong U'_1 \) and \( U_2 \cong U'_2 \) as \( A \)-modules, if and only if (b) there exists an invertible \( \alpha \in \text{End}_A(U) \) so that \( f'_1 = \alpha f_1 \alpha^{-1} \), if and only if (c) as \( \text{End}_A(U) \)-modules, \( \text{End}_A(U)f_1 \cong \text{End}_A(U)f'_1 \) and \( \text{End}_A(U)f_2 \cong \text{End}_A(U)f'_2 \).