1. (8 points) Give a proof of the following result by following the suggested steps.

**THEOREM.** Let $E \supset F$ be a field extension of finite degree and let $A$ be an $F$-algebra. Let $U$ and $V$ be $A$-modules. Then

$$E \otimes_F \text{Hom}_A(U,V) \cong \text{Hom}_{E \otimes_F A}(E \otimes_F U, E \otimes_F V)$$

via an isomorphism $\lambda \otimes f \mapsto (\mu \otimes u \mapsto \lambda \mu \otimes f(u))$.

(a) Verify that there is indeed a homomorphism as indicated.
Solution: We verify that the specification $\mu \otimes f(u) \mapsto \lambda \mu \otimes f(u)$ is balanced for $F$. This is because if $x \in F$ then $\mu x \otimes f(u) = \lambda \mu x \otimes f(u) = \lambda \mu \otimes f(xu)$ and this is what $\mu \otimes f$ is sent to. We also check that the assignment on $\lambda \otimes f$ is $F$-balanced by showing similarly that $\lambda x \otimes f$ is sent to the same mapping as $\lambda \otimes xf$.

(b) Let $x_1, \ldots, x_n$ be a basis for $E$ as an $F$-vector space. Show that for any $F$-vector space $M$, each element of $E \otimes_F M$ can be written uniquely in the form $\sum_{i=1}^n x_i \otimes m_i$ with $m_i \in M$.
Solution: Each element of $E \otimes_F M$ can be written in the form $\sum_{i=1}^n \lambda_i x_i \otimes u_i$ with $u_i \in M$ and $\lambda_i \in F$. Because $E$ is free as an $F$-module with the given basis, each term is in the sum has a unique value. Since $\lambda_i x_i \otimes u_i = x_i \otimes \lambda_i u_i$ and $x_i \otimes M \cong M$, putting $m_i = \lambda_i u_i$ we obtain a unique expression for this term as $x_i \otimes m_i$.

(c) Show that if an element $\sum_{i=1}^n x_i \otimes f_i \in E \otimes_F \text{Hom}_A(U,V)$ maps to 0 then $\sum_{i=1}^n x_i \otimes f_i(u) = 0$ for all $u \in U$. Deduce that the homomorphism is injective.
Solution: If $\sum_{i=1}^n x_i \otimes f_i$ maps to the zero mapping then the effect of the image map on $1 \otimes u$ is $\sum_{i=1}^n x_i \otimes f_i(u)$, and this is zero, for all $u$. By the uniqueness from (b) we deduce that $x_i \otimes f_i(u) = 0$ always, which implies $x_i \otimes f_i = 0$ and hence that $\sum_{i=1}^n x_i \otimes f_i = 0$. Thus the homomorphism is injective.

(d) Show that the homomorphism is surjective as follows: given an $E \otimes_F A$-module homomorphism $g : E \otimes_F U \rightarrow E \otimes_F V$, write $g(1 \otimes_F u) = \sum_{i=1}^n x_i \otimes f_i(u)$ for some elements $f_i(u) \in V$. Show that this defines $A$-module homomorphisms $f_i : U \rightarrow V$. Show that $g$ is the image of $\sum_{i=1}^n x_i \otimes f_i$.

Solution: If $a \in A$ we have that $\sum_{i=1}^n x_i \otimes f_i(au) = g(1 \otimes_F au) = (1 \otimes_F g)(1 \otimes_F u) = (1 \otimes_F a) \sum_{i=1}^n x_i \otimes f_i(u) = \sum_{i=1}^n x_i \otimes af_i(u)$. By uniqueness of expression we deduce $f_i(au) = af_i(u)$ always and $f_i$ is an $A$-module homomorphism. Now the image of $\sum_{i=1}^n x_i \otimes f_i$ equals $g$ on elements $1 \otimes_F u$, and hence equals $g$ since $g$ is $E$-linear.

2. (5 points) The anti-automorphism of $S_F(n,r)$ used in defining the dual of a representation of the Schur algebra was defined as sending an endomorphism of $E \otimes r$ to its transpose with respect to the standard bilinear form on $E \otimes r$. Compute the effect of
this antiautomorphism on the basis elements $\xi_{i,j}$ of $S_F(n, r)$ constructed as the duals of the monomial functions $e_{i,j}$. Solution: We have seen in class that $\xi_{a,b}(e_{b}) = \sum_{k \in \text{Stab}_{S_n}(b)} e_{k}$. This means that the matrix of $\xi_{a,b}$ has a 1 in every position $(a \pi, b \pi)$. Applying the antiautomorphism we get an element that acts by the transpose matrix, with a 1 in every position $(b \pi, a \pi)$ and this is the matrix of $\xi_{b,a}$. Thus $\xi_{i,j}$ is exchanged with $\xi_{j,i}$.

3. (5 points) For any finite dimensional representation $V$ of a group $G$ we can construct another representation $V^*$ whose representation space is $\text{Hom}_F(V, F)$ and where $g \in G$ acts on a linear map $f : V \to F$ to give $gf$, where $gf(v) = f(g^{-1}v)$. Suppose that $F$ is infinite and $V$ is a polynomial representation of $GL_n(F)$. Show that $V^*$ is polynomial if and only if $GL_n(F)$ acts trivially on $V$.

Solution: If $g = \text{diag}(t_1, \ldots , t_n)$ then it acts on each weight space $V^\alpha$ as $t_1^{\alpha_1} \cdots t_n^{\alpha_n}$. On $V^*$ the $-\alpha$ weight space is thus nonzero, so that if $V^*$ is polynomial then both $\alpha$ and $-\alpha$ must be non-negative, and hence zero. Thus $V = V^0$ is the zero weight space and the diagonal subgroup acts trivially on $V$. From this it follows that each diagonalizable element acts trivially on $V$. Since these elements are dense in $GL_n(F)$, the whole group must act trivially on $V$. Thus if $V$ is polynomial it must be trivial. Conversely, if $V$ has trivial action then so does $V^*$ so that $V^*$ is polynomial.

4. (5 points) Show that the simple $S_F(n, r)$-modules are self-dual.

Solution: The formal character of a representation and its dual $V^\circ$ are always the same. Since the simple modules are determined by their formal characters, they are self-dual. This is because the antiautomorphism of $S_F(n, r)$ fixes $\xi_{\alpha}$ and so the $\alpha$-weight spaces of $V$ and $V^\circ$ pair in a non-degenerate fashion, and hence have the same dimension.

5. (5 points) In the situation where we have an algebra $B$ containing an idempotent $e$ and a Schur functor $f : B$-mod $\to eBe$-mod, show that the left adjoint and the right adjoint functors of $f$ need not be naturally isomorphic. The left adjoint is $W \mapsto Be \otimes eBe W$ and the right adjoint is $W \mapsto \text{Hom}_{eBe}(eB, W)$.

Solution: Let $A = FS_2$ where $F = \mathbb{F}_2$. We have seen that the regular representation is an indecomposable module with structure $A = \mathbb{F}_2$ and the Schur algebra $B = S_F(2, 2) = \text{End}_A(F \oplus A)$ has dimension 3. Let $e \in B$ be projection onto the second summand. We have seen $eBe \cong A$, and that $e$ and $1 - e$ are primitive orthogonal idempotents corresponding to simple $B$-modules $\beta$ and $\alpha$, so that $e\beta \neq 0$ and $e\alpha = 0$, $(1 - e)\beta = 0$ and $(1 - e)\alpha \neq 0$. Let $W$ be the simple $A$-module $F$. If $S$ is a simple $B$-module then $\text{Hom}_B(Be \otimes eBe W, S) \cong \Hom_{eBe}(W, \text{Hom}_B(Be, S))$. The left adjoint is $W \mapsto Be \otimes eBe W$ and the right adjoint is $W \mapsto \text{Hom}_{eBe}(eB, W)$. We have also calculated that the projective $B$-module $Be$ is uniserial with top composition factor $\beta$, so the latter group is non-zero when $S \cong \beta$, zero when $S = \alpha$. As a right $A$-module, $Be \cong F \oplus A$, so $Be \otimes eBe W$ has dimension 2. It follows that $Be \otimes eBe W$ is uniserial with top composition factor $\beta$, bottom composition factor $\alpha$. By a similar argument, $\text{Hom}_B(S, \text{Hom}_A(eB, W)) \cong \text{Hom}_A(eB \otimes B S, W)$ is nonzero if $S = \beta$, zero if $S = \alpha$. This shows that $Be \otimes eBe W$ is not isomorphic to $\text{Hom}_{eBe}(eB, W)$. 2