1. Let $P_S$ be an indecomposable projective module for a finite dimensional algebra over a field. Show that every non-zero homomorphic image of $P_S$

(a) has a unique maximal submodule,
(b) is indecomposable, and
(c) has $P_S$ as its projective cover.

2. Let $A$ be a finite dimensional algebra.

(a) Show that if $f : U \rightarrow V$ is a homomorphism of $A$-modules for which the restriction $f|_{\text{Soc } U} : \text{Soc } U \rightarrow V$ is one-to-one then $f$ is one-to-one.

(b) Show that the injective envelope of $A/\text{Rad } A$ has the same dimension as $A$. (You may assume question 7 from homework 3.)

(c) Show that if $A/\text{Rad } A \cong \text{Soc } A$ as left $A$ modules then the left regular representation $A_A$ is injective as a left $A$ module; and also that the right regular representation $A_A$ is injective as a right $A$ module. (An algebra satisfying this condition is called self-injective)

(d) Give an example of a self-injective algebra that is not semisimple.

3. Let $A$ be a finite dimensional algebra and let $U$ be an $A$-module.

(a) Prove that if $U$ is indecomposable then $\text{Rad}_A(U,U) = \text{Rad}_A^2(U,U)$.

(b) Find an example of an algebra $A$ and a module $U$ for which $\text{Rad}_A(U,U) \neq \text{Rad}_A^2(U,U)$.

4. Let $A$ be a finite dimensional commutative algebra. Show that $A$ is a finite product of commutative local algebras. (The product of algebras $A$ and $B$ is often written as a direct sum $A \oplus B = \{(a,b) \mid a \in A, b \in B\}$.)

5. Let $\alpha : U \rightarrow V_1 \oplus V_2$ be a homomorphism of finite dimensional $A$-modules where $A$ is a finite dimensional algebra over a field, so that we can write

$\alpha = \begin{bmatrix} \alpha_1 \\ \alpha_2 \end{bmatrix}$

where $\alpha_i = p_i \circ \alpha : U \rightarrow V_i$ are the component maps of $\alpha$, the $p_i : V_1 \oplus V_2 \rightarrow V_i$ being the projections with respect to the direct sum decomposition. Suppose that $U$ is indecomposable, so that $\text{End}_A(U)$ is a local ring.

(a) Show that if $\alpha_1$ is split mono then $\alpha$ is split mono.
(b) Show that if $\alpha$ is split mono then one of $\alpha_1$ and $\alpha_2$ is split mono.

(c) Show that if $\alpha$ is an irreducible morphism then neither of $\alpha_1, \alpha_2$ is split epi.

(d) Show that if $\alpha$ is an irreducible morphism then each of $\alpha_1$ and $\alpha_2$ is an irreducible morphism.

6. Let $A$ be a finite dimensional $K$-algebra such that $\text{Rad}_A^m(-, -) = 0$ for some $m \geq 1$. Prove that any nonzero nonisomorphism between indecomposable modules in $A$-$\text{mod}$ is a sum of compositions of irreducible morphisms.

7. In this question, describe modules by showing their composition factors, in such a way that we can also see the composition factors if their radical and socle series. A diagrammatic notation (as done in class) is sufficient to achieve this. Let $P$ be the poset with four elements $\{1, 2, 3, 4\}$ and partial order $1 < 2 < 4, 1 < 3 < 4$ so that the Hasse diagram is:

![Hasse diagram]

We consider representations of this poset over $\mathbb{Q}$, namely, modules for the category algebra $\mathbb{Q}P$ of $P$ regarded as a category $\mathcal{P}$, over $\mathbb{Q}$, which are the same thing as functors from $\mathcal{P}$ to $\mathbb{Q}$-vector spaces.

(a) Describe (in the sense just explained) the four indecomposable projective representations, and also the four indecomposable injective representations.

(b) For each of the four simple modules $S$, compute $DT_r(S)$.

(c) Write down all almost split sequences that have as a middle term a module that is both projective and injective.

(d) Complete the calculation of the Auslander-Reiten quiver of $\mathbb{Q}P$, giving a justification for each calculation made.

8. Find the error in the following argument (perhaps showing by example what is wrong) and then give an example as requested at the very end:

**Theorem 0.1.** Let $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ be an almost split sequence. If $V = V_1 \oplus V_2$ is the direct sum of two non-zero submodules then the restriction of $\beta$ to each of $V_1$ and $V_2$ is one-to-one.
Proof. Let \( V = V_1 \oplus V_2 \) and let \( p_i : V \to V_i \) be projection and \( \iota_i : V_i \to V \) be inclusion with respect to this direct sum decomposition, \( i = 1, 2 \). Suppose one of the component maps \( \beta \circ \iota_i = \beta|_{V_i} : V_i \to W \) is epi. Then it is not split epi, because otherwise \( \beta \) would be split epi. Consider the commutative diagram with exact rows

\[
\begin{array}{cccccc}
0 & \to & U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \to & 0 \\
0 & \to & \text{Ker } \beta|_{V_i} & \xrightarrow{\iota_i} & V_i & \xrightarrow{\beta|_{V_i}} & W & \to & 0 \\
\end{array}
\]

Because \( \iota_i \) is split by \( p_i \), the restriction \( \iota_i|_{\text{Ker } \beta|_{V_i}} \) is split by \( p_i|_U \), so \( \iota_i|_{\text{Ker } \beta|_{V_i}} \) is split mono. Now \( U \) is indecomposable, so \( \text{Ker } \beta|_{V_i} \) is either isomorphic to \( U \) or is 0. In the first case \( \iota_i \) is an isomorphism, so \( V \) does not have two summands. In the second case \( V_i \cong W \) and \( \beta \) is split epi. Both of these are contradictions, so each restriction \( \beta|_{V_i} \) is one-to-one. 

Give an example of an almost split sequence where the middle term has two direct summands and the restriction of \( \beta \) to one of them is epi.