Math 8245 **Homework 4, due Monday 10 November** **PJW**

GAP questions:

1. (Question 2 from page 114 of Johnson's book.) Prove in detail that in enumerating cosets for the presentation

$$T_n = \langle x, y \mid x^n y^{n+1}, x^{n+1} y^{n+2} \rangle, \quad n \in \mathbb{N},$$

at least $n+1$ symbols are needed. Can you enlarge this lower bound?

2. Let g denote the group with presentation given in question 1. When $n = 3$ the command
`gap> CosetTable(g,Subgroup(g,[]));`
succeeds in showing that there is only one coset, but when $n = 20$ it fails.
   (i) Find the least value of n for which this command fails to enumerate the cosets, without increasing the default number of cosets allowed.
   (ii) Investigate what is going on and give an explanation, given that it is possible to show by coset enumeration that there is only one coset by introducing far fewer cosets than the default maximum. Does GAP simply make poor choices of cosets which it introduces?

Theory questions: These are taken from the end of section 4, and have the same numbering as the questions there.

2. Let G be the non-abelian group of order 21:

$$G = \langle x, y \mid x^7 = y^3 = 1, yxy^{-1} = x^2 \rangle.$$

Show that G has 5 conjugacy classes, and find its character table.

5. Three-suffix tensors have components $\tau_{ijk} \in \mathbb{R}$ where $i, j, k \in \{1, 2, 3\}$, and form a vector space V of dimension 27 over $\mathbb{R}$. The symmetric group S_3 acts on V by permuting the suffixes. Decompose the space V as a direct sum of simple representations of S_3, giving the mutiplicities of each simple representation. [Observe that V is a permutation representation.]
Give also the decomposition of V as a direct sum of three subspaces consisting of tensors with different symmetry properties under S_3. What are the dimensions of these subspaces?

Extra questions:

3. Find the character table of the following group of order 36:

$$G = \langle a, b, c \mid a^3 = b^3 = c^4 = 1, ab = ba, cac^{-1} = b, cbc^{-1} = a^2 \rangle.$$

[It follows from these relations that $\langle a, b \rangle$ is a normal subgroup of G of order 9.]