Math 8246 **Homework 1, due Monday February 9, 2004** **PJW**

Hand in solutions to any 5 of the 10 questions

1. (Section 8) Prove that if G is any finite group then the only idempotents in the integral group ring $\mathbb{Z}G$ are 0 and 1.
   [If e is idempotent consider the rank of the free abelian group $\mathbb{Z}Ge$ and also its image under the homomorphism $\mathbb{Z}G \to \mathbb{F}_pG$ for each prime p dividing $|G|$, which is a projective $\mathbb{F}_pG$-module. Show that $\text{rank}_{\mathbb{Z}} \mathbb{Z}Ge$ is divisible by $|G|$. Deduce from this that if $e \neq 0$ then $e = 1$.]

2. (Section 8) (a) Let $H = C_2 \times C_2$ and let k be a field of characteristic 2. Show that $(IH)^2$ is a one-dimensional space spanned by $\sum_{h \in H} h$.
   (b) Let $G = A_4 = (C_2 \times C_2) \rtimes C_3$ and let $\mathbb{F}_4$ be the field with four elements. Compute the radical series of each of the three indecomposable projectives for $\mathbb{F}_4 A_4$ and identify each of the quotients
   $$\text{Rad}^n P_S / \text{Rad}^{n+1} P_S.$$
   Now do the same for the socle series. Hence determine the Cartan matrix of $\mathbb{F}_4 A_4$.
   [Start by observing that $\mathbb{F}_4 A_4$ has 3 simple modules, all of dimension 1, which one might denote by 1, ω and ω^2.]
   (c) Now consider $\mathbb{F}_2 A_4$ where $\mathbb{F}_2$ is the field with two elements. Prove that the 2-dimensional $\mathbb{F}_2$-vector space on which a generator of C_3 acts via $\begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$ is a simple $\mathbb{F}_2 C_3$-module. Calculate the radical and socle series for each of the two indecomposable projective modules for $\mathbb{F}_2 A_4$ and hence determine the Cartan matrix of $\mathbb{F}_2 A_4$.

17. (Section 6) (a) Let G be any group and $IG \subset \mathbb{Z}G$ the augmentation ideal over $\mathbb{Z}$. Prove that $IG/(IG)^2 \cong G/G'$ as abelian groups.
   [Consider the homomorphism of abelian groups $IG \to G/G'$ given by $g - 1 \mapsto gG'$. Use the formula $ab - 1 = (a-1) + (b-1) + (a-1)(b-1)$ to show that $(IG)^2$ is contained in the kernel, and that the homomorphism $G/G' \to IG/(IG)^2$ given by $gG' \mapsto g - 1 + (IG)^2$ is well defined.]
   (b) If now R is any commutative ring with 1 and $IG \subset RG$ is the augmentation ideal, show that $IH/(IH)^2 \cong R \otimes_{\mathbb{Z}} G/G'$ as R-modules.

3. (Section 8) Let $G = H \rtimes K$ where H is a p-group, K is a p'-group, and let k be a field of characteristic p. Regard kH as a kG-module via its isomorphism with P_k, so H acts as usual and K acts by conjugation.
   (a) Show that for each n, $(IH)^n$ is a kG-submodule of kH, and that $(IH)^n/(IH)^{n+1}$ is a kG-module on which H acts trivially.
   (b) Show that
   $$P_k = kH \supseteq IH \supseteq (IH)^2 \supseteq (IH)^3 \cdots$$

is the radical series of P_k as a kG-module.
(c) Show that there is a map

$$IH/(IH)^2 \otimes_k (IH)^n/(IH)^{n+1} \to (IH)^{n+1}/(IH)^{n+2}$$
$$x + (IH)^2 \otimes y + (IH)^{n+1} \mapsto xy + (IH)^{n+2}$$

which is a map of kG-modules. Deduce that $(IH)^n/(IH)^{n+1}$ is a homomorphic image of $(IH/(IH)^2)^{\otimes n}$.
(d) Show that the abelianization H/H' becomes a $\mathbb{Z}G$-module under the action $g \cdot xH' = gxg^{-1}H'$. Show that the isomorphism $IH/(IH)^2 \to k \otimes_{\mathbb{Z}} H/H'$ specified by $(x-1)+(IH)^2 \mapsto 1 \otimes xH'$ of Section 6 Exercise 17 is an isomorphism of kG-modules.

15. (Section 6) The generalized quaternion group of order 2^n has a presentation

$$Q_{2^n} = \langle x, y \mid x^{2^{n-1}} = 1,\ y^2 = x^{2^{n-2}},\ yxy^{-1} = x^{-1} \rangle.$$

Let k be a field of characteristic 2. Show that when $r \geq 1$ each power $(IQ_{2^n})^r$ of the augmentation ideal is spanned modulo $(IQ_{2^n})^{r+1}$ by $(x-1)^r$ and $(x-1)^{r-1}(y-1)$. Hence calculate the Loewy length of kQ_{2^n}.

4. (Section 8) The group $SL(2,3)$ is isomorphic to the semidirect product $Q_8 \rtimes C_3$ where the cyclic group C_3 acts on $Q_8 = \{\pm 1, \pm i, \pm j, \pm k\}$ by cycling the three generators i, j and k. Assuming this structure, compute the radical series of each of the three indecomposable projectives for $\mathbb{F}_4 SL(2,3)$ and identify each of the quotients

$$\operatorname{Rad}^n P_S / \operatorname{Rad}^{n+1} P_S.$$

[Use Section 6 Exercise 15.]

5. (Section 8) Let $G = P \rtimes S_3$ be a group which is the semidirect product of a 2-group P and the symmetric group of degree 3. (Examples of such groups are $S_4 = V \rtimes S_3$ where $V = \langle (1,2)(3,4), (1,3)(2,4) \rangle$, and $GL(2,3) \cong Q_8 \rtimes S_3$ where Q_8 is the quaternion group of order 8.)
(a) Let k be a field of characteristic 2. Show that kG has two non-isomorphic simple modules.
(b) Let $e_1, e_2, e_3 \in \mathbb{F}_4 S_3$ be the orthogonal idempotents which appeared in Example 7.5. Show that each e_i is primitive in $\mathbb{F}_4 G$ and that $\dim \mathbb{F}_4 G e_i = 2|P|$ for all i.
[Use the fact that the $\mathbb{F}_4 G e_i$ are projective modules.]
(c) Show that if $e_1 = () + (1,2,3) + (1,3,2)$ then $\mathbb{F}_4 S_4 e_1$ is the projective cover of the trivial module and that $\mathbb{F}_4 S_4 e_2$ and $\mathbb{F}_4 S_4 e_3$ are isomorphic, being copies of the projective cover of a 2-dimensional module.
(d) Show that $\mathbb{F}_4 G e_i \cong \mathbb{F}_4 \langle (1,2,3) \rangle e_i \uparrow_{\langle (1,2,3) \rangle}^{G}$ for each i.

6. (Section 8) Let A be a finite-dimensional algebra over a field k, and let A_A be the right regular representation of A. The vector space dual $(A_A)^* = \operatorname{Hom}_k(A_A, k)$ becomes a left A-module via the action $(af)(b) = f(ba)$ where $a \in A$, $b \in A_A$ and $f \in (A_A)^*$. Prove that the following two statements are equivalent:
   (a) $(A_A)^* \cong {}_AA$ as left A-modules.
   (b) There is a non-degenerate associative bilinear pairing $A \times A \to k$.
   An algebra satisfying these conditions is called a *Frobenius algebra*. Prove that, for a Frobenius algebra, projective and injective modules are the same thing.

7. (Section 8) Let A be a finite-dimensional algebra over a field k and suppose that the left regular representation ${}_AA$ is injective. Show that every projective module is injective and that every injective module is projective.

8. (Section 8) Let S and T be simple kG-modules, with projective covers P_S and P_T, where k is an algebraically closed field.
   (a) For each n prove that

$$\begin{aligned}\operatorname{Hom}_{kG}(P_T, \operatorname{Soc}^n P_S) &= \operatorname{Hom}_{kG}(P_T / \operatorname{Rad}^n P_T, \operatorname{Soc}^n P_S)\\ &= \operatorname{Hom}_{kG}(P_T / \operatorname{Rad}^n P_T, P_S).\end{aligned}$$

   (b) Deduce Landrock's theorem: the multiplicity of T in the nth socle layer of P_S equals the multiplicity of S in the nth radical layer of P_T.
   (c) Use Exercise 6 of Section 6 to show that these multiplicities equal the multiplicity of T^* in the nth radical layer of P_{S^*}, and also the multiplicity of S^* in the nth socle layer of P_{T^*}.