Math 8202 Homework 9 PJW

Date due: April 3, 2006.

Hand in only the starred questions.

Section 13.4, page 525 3*, 4.

Section 13.5, page 531 5*, 6*, 7, 8, 10*.

F*. (Fall 2002 qn. 5, part (a)) Let \( k \) be a field of characteristic \( p > 0 \), and \( K = k(t) \) where \( t \) is an element transcendental over \( k \). Show that \( X^p - t \) is irreducible in \( K[X] \).

G*. (Fall 2001, qn. 6) (10%) Let \( \mathbb{F}_{p^k} \) be the field with \( p^k \) elements, where \( p \) is prime.
   (a) Show that \( x^4 + 1 \in \mathbb{F}_p[x] \) has a root in \( \mathbb{F}_{p^2} \).
   (b) Deduce that \( x^4 + 1 \) is reducible in \( \mathbb{F}_p[x] \). For which values of \( p \) does a linear factor exist in \( \mathbb{F}_p[x] \)?
   [You may assume standard facts about finite fields.]

H. (Fall 2000, qn. 5)(12%) Let \( K \supseteq k \) be a field extension and \( f \in k[X] \) an irreducible polynomial of degree relatively prime to the degree of the field extension \( [K : k] \). Show that \( f \) is irreducible in \( K[X] \).

I. (Fall 2000, qn. 6)(15%) a) (8) Let \( K \supseteq k \) be a field extension of prime degree, and let \( a \in K \) be an element which does not lie in \( k \). Considering \( K \) as a vector space over \( k \), let \( m_a : K \to K \) be the \( k \)-linear mapping specified by \( m_a(x) = ax \). Prove that the characteristic polynomial of \( m_a \) is irreducible.
   b) (7) Let \( \alpha \) be a root of \( X^3 - X + 1 \) in \( \mathbb{F}_{27} \). Find the minimal polynomial of \( \alpha^4 \) over \( \mathbb{F}_3 \).
   [Here \( \mathbb{F}_{27} \) and \( \mathbb{F}_3 \) denote fields with 27 and 3 elements, respectively. You may assume that \( X^3 - X + 1 \) is irreducible in \( \mathbb{F}_3[X] \).]