Math 8245 **Theory Homework 1** **PJW**
Date due: September 18, 2006

Some general questions

1. Let $N \lhd G = H \times K$. Prove that either N is abelian or N intersects one of the factors H or K nontrivially.

2. If $H \leq L \leq G$ and $N \lhd G$ show that the equations $HN = LN$ and $H \cap N = L \cap N$ imply that $H = L$.

3. a) (The modular law) Let H, K, and L be subgroups of G with $H \subseteq L$. Show that

$$HK \cap L = H(K \cap L).$$

   b) Suppose we remove the requirement in a) that $H \subseteq L$. Give an example to show that the conclusion need not hold.

4. Let G be a finite group with a normal subgroup H such that $(|H|, |G:H|) = 1$. Show that H is the unique subgroup of G having order $|H|$.
   [Hint: If K is another such subgroup, what happens to K in G/H?]

Semidirect and wreath products

5. Let G be a group, and consider the usual homomorphism $\theta : G \to \operatorname{Aut} G$ where $\theta(g)(x) = gxg^{-1}$, so $\theta(g)$ is conjugation by g. Using θ we may form the semidirect product $G \rtimes G$. Show that $G \rtimes G \cong G \times G$.
   [Hint: Look for a subgroup of $G \times G$ which acts on G via θ.]

6. Let S_G be the group of all permutations of G (the symmetric group on G), and observe that $\operatorname{Aut}(G)$ is a subgroup of S_G. Let $\lambda : G \to S_G$ be the homomorphism given by the left regular representation of G, so for each $g \in G$, $\lambda(g)$ is the permutation of G given by $\lambda(g)(x) = gx$, and let $\rho : G \to S_G$ be the homomorphism given by the right regular representation of G, so for each $g \in G$, $\rho(g)$ is the permutation of G given by $\rho(g)(x) = xg^{-1}$.
   (a) Show that $\langle \lambda(G), \operatorname{Aut}(G) \rangle = \langle \rho(G), \operatorname{Aut}(G) \rangle$ as subgroups of S_G, and they have the form $G \rtimes \operatorname{Aut}(G)$ (a group known as the *holomorph* of G).
   (b) Show that $N_{S_G}(\lambda(G)) = \langle \lambda(G), \operatorname{Aut}(G) \rangle$.
   (c) Deduce (for example) that

$$N_{S_8}(\langle (1,2)(3,4)(5,6)(7,8), (1,3)(2,4)(5,7)(6,8), (1,5)(2,6)(3,7)(4,8) \rangle) \cong (C_2 \times C_2 \times C_2) \rtimes GL(3,2).$$

   [This question seems fairly hard, and you may wish to proceed using the following steps.

a) Establish the formula $\alpha\lambda(g)\alpha^{-1} = \lambda(\alpha(g))$ for all $\alpha \in S_G$ and $g \in G$.
b) Any $\beta \in N_{S_G}(\lambda(G))$ can be written $\beta = \lambda(g)\beta'$ for some $g \in G$, where $\beta'(1) = 1$.
c) Given $\gamma \in N_{S_G}(\lambda(G))$ there exists $\alpha \in \mathrm{Aut}(G)$ with $\gamma\lambda(g)\gamma^{-1} = \lambda(\alpha(g))$ for all $g \in G$. Deduce that $\alpha^{-1}\gamma \in C_{S_G}(\lambda(G))$.
d) Show that if $\delta \in C_{S_G}(\lambda(G))$ and $\delta(1) = 1$, then δ is the identity permutation of G.
e) Put the previous pieces together!]

7. Prove that the standard wreath product $\mathbb{Z}\wr\mathbb{Z}$ is finitely generated but has a non-finitely generated subgroup.

8. Prove that the standard wreath product $C_2 \wr C_2$ is isomorphic to D_8.

9. Let
$$G = \{\begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{Z}/2\mathbb{Z}\} \subseteq GL(4, 2).$$
Show that $G \cong C_2 \wr (C_2 \times C_2)$ where the $C_2 \times C_2$ acts regularly on a set of size 4. [First show that $G = N \rtimes H$ where N is the subgroup specified by $a = f = 0$ and H is the subgroup specifed by $b = c = d = e = 0$.]