Math 8245 **Homework 6** **PJW**

Date due: December 11, 2006. Either hand it to me in class or put it in my mailbox by 3:30.

1. Let $q = p^n$ be a prime power and let $\mathbb{F}_q$ be the field with q elements. Let G be the group of permutations of the set $\mathbb{F}_q$ which is generated by all the permutations $a_y : \mathbb{F}_q \to \mathbb{F}_q$ and $m_z : \mathbb{F}_q \to \mathbb{F}_q$ defined by $a_y(x) = x + y$, $m_z(x) = xz$, where $y \in \mathbb{F}_q$ and $z \in \mathbb{F}_q^\times$.
   (i) Show that $G \cong \mathbb{F}_q \rtimes \mathbb{F}_q^\times$.
   (ii) Show that G is isomorphic to the group of matrices $\{\begin{pmatrix} y & z \\ 0 & 1 \end{pmatrix} \mid y \in \mathbb{F}_q^\times,\ z \in \mathbb{F}_q\}$.
   (iii) Show that G is sharply 2-transitive in its action on $\mathbb{F}_q$.

2. (page 220, 9.47) Show that the four-group $V = \{(), (1,2)(3,4), (1,3)(2,4), (1,4)(2,3)\}$ has no transitive extension. [Hint: If $h \in S_5$ has order 5, then $\langle V, h\rangle \supseteq A_5$.]

3. Only do this question if we have said something about primitive actions and regular normal subgroups in class.
   (a) Let G be a finite group which acts faithfully and primitively on a set X, and let p be a prime number with the following properties:
   (i) $p \mid |X|$, and p divides $|G|$ exactly once.
   (ii) G is generated by elements of order p.
   Show that G is simple.
   (b) Assume from calculations in GAP that M_{11} is sharply 4-transitive and is generated by a and bab^{-1} where

$$a = (1,2,3,4,5,6,7,8,9,10,11) \qquad \text{and} \qquad b = (3,7,11,8)(4,10,5,6).$$

   Show that M_{11} is simple.

The next questions are all about Frobenius groups. One definition is that a finite group G is called a *Frobenius group* if and only if it has a faithful transitive action on a set X so that a) no non-identity element of G has more than one fixed point, and b) there is at least one non-identity element of G with a fixed point. We will see in questions 6, 7 and 8 an equivalent definition of a Frobenius group.

Frobenius proved that $H = \{g \in G \mid g \text{ has no fixed points}\} \cup \{1\}$ is a normal subgroup of G under the hypothesis that G is a Frobenius group. The subgroup H is called the *Frobenius kernel*. The proof is by character theory and is outside the scope of the course. The difficult part is in showing that H is a subgroup – that H is normal is then easy. Please assume the theorem of Frobenius in what follows, and also the notation that G is a Frobenius group with kernel H. In Rotman's book an argument is presented which proves Frobenius' theorem in the special case that G is a sharply 2-transtive group, but you do not need to read this.

4. Show that the Frobenius kernel H acts regularly on X (transitively, with identity stabilizers), so that in particular $|X| = |H|$.
5. Show that if x is any element of X, putting $C = G_x$ we have $G = H \rtimes C$. (The subgroup C is called the *Frobenius complement.*)
6. Show that for every non-identity element $c \in C$, if $h \in H$ with $chc^{-1} = h$ then $h = 1$. (We say C acts *fixed-point freely* on $H^\# := H - \{1\}$.)
7. Let A be a finite group and suppose that B is a group of automorphisms of A with the property that for all $1 \neq b \in B$, b acts fixed-point freely on $A^\# := A - \{1\}$. Show that $A \rtimes B$ is a Frobenius group. [Identify A with the set X on which $A \rtimes B$ is to act. Construct a Frobenius action on X. It may help to be guided by the action of the group in question 1.]
8. In a Frobenius group G, show that if $1 \neq h \in H$ then $C \cap {}^hC = 1$.
9. Show that G is the disjoint union of H and the sets $({}^hC)^\#$ with $h \in H$.

Here are some (more) examples of Frobenius groups: A_4, D_{2n} when n is odd, $C_2 \times C_2 \times C_2 \rtimes C_7$ with the C_7 acting as the Sylow 7-subgroup of $GL(3, 2)$.

10. Let G act on X 2-transitively. Show that the action is a Frobenius action (i.e. G is a Frobenius group) if and only if the action is sharply 2-transitive. Is it necessarily the case that every Frobenius group acts 2-transitively on X?

It is the case that in every Frobenius complement, Sylow subgroups are either cyclic or generalized quaternion (fairly easy), and that Frobenius kernels are always nilpotent (this appears in the thesis of J.G. Thompson which is sometimes held to signal the start of the activity leading to the classification of finite simple groups).

11. Using GAP, show that the stabilizer of 3 points in M_{12} is a Frobenius group with structure $(C_3 \times C_3) \rtimes Q_8$. Show that $GL(2, 3)$ has only one conjugacy class of subgroups isomorphic to Q_8 (you may quote prior homework). Deduce that any two Frobenius groups with structure $(C_3 \times C_3) \rtimes Q_8$ are conjugate in $(C_3 \times C_3) \rtimes GL(2, 3)$ and hence are isomorphic.
12. Suppose that n players enter a Scrabble tournament, in which games are played with 4 people playing in each game. What is the smallest number of games that must be played so that each pair of players plays against each other in at least one of the games? Answer this question when (a) $n = 5$, (b) $n = 6$, (c) $n = 16$.
13. Given a $t - (v, k, \lambda)$ design $\mathcal{D}$ on a set S, and given $p \in S$, the *residual design* $\mathcal{D}^p$ has set $S - \{p\}$ and as blocks the blocks of $\mathcal{D}$ which do not contain p. This is a $(t-1) - (v-1, k, \lambda')$ design for some parameter λ'. Find an expression for λ' in terms of t, v, k and λ.

Extra questions: do not hand in.

14. (page 183, 9.11) Let G act k-transitively on a set X. We define an action on the product set X^k by $g(x_1, \ldots, x_k) = (gx_1, \ldots, gx_k)$. Show that if $(x_1, \ldots, x_k)$ and $(y_1, \ldots, y_k)$ are two lists of k elements of X such that the elements in each list are distinct then the stabilizers $G_{(x_1,\ldots,x_k)}$ and $G_{(y_1,\ldots,y_k)}$ are conjugate subgroups of G.

15. (page 227, 9.57) Let X be an m-dimensional vector space over $\mathbb{F}_2$ and let B be the family of all affine planes of X (that is, subsets of X of the form $P + v$ where P is a 2-dimensional linear subspace of X and $v \in X$). Show that if $m \geq 3$, then (X, B) is a Steiner system of type $S(3, 4, 2^m)$. [Hint: Three distinct points cannot be collinear.]