1. Let $G := \langle x \rangle$ be cyclic of order 15. Find the order of the element $x^6$ in $G$.

2. Find the remainder of $9^{1573}$ when divided by 11.

3. Let $G$ be a group of order 36. If $G$ has an element $a \in G$ such that $a^{12} \neq 1$ and $a^{18} \neq 1$, show that $G$ is cyclic.

4. Show that the mapping $G \to G$ specified by $x \to x^{-1}$ is a group homomorphism if and only if $G$ is abelian.

5. Let $C_7 = \langle x \rangle$, $C_6 = \langle y \rangle$, $C_2 = \langle z \rangle$ be cyclic groups generated by elements $x, y, z$ of orders 7, 6 and 2 respectively.
   (a) Is $\phi : C_6 \to C_2$ given by $\phi(y) = z$ a homomorphism?
   (b) Is $\phi : C_7 \to C_2$ given by $\phi(x) = z$ a homomorphism?

6. (a) Find all possible homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$.
   (b) Find all possible homomorphisms from $\mathbb{Z}$ onto $\mathbb{Z}$.

7. The quaternion group of order 8 is the set
   $$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$
   with multiplication given by the rules
   $$i^2 = j^2 = k^2 = -1,$$
   $$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j,$$
   $$(−1)x = x(−1) = −x, \quad 1x = x1 = x$$
   for all $x$ (with the understanding that $−(−x) = x$). Write out a list of the orders of the elements of $Q_8$. Make a complete list of all the subgroups of $Q_8$ and determine which of them are normal. Show that the factor group $Q_8/\{1, -1\}$ is isomorphic to $C_2 \times C_2$.

8. Let $G := C_{15} = \langle x \rangle$ be cyclic of order 15. Find the order of the factor group $Q := G/\langle x^6 \rangle$. Find the order of the element $x^3\langle x^6 \rangle$ in $Q$. Find the order of the element $x^2\langle x^6 \rangle$ in $Q$.

9. (page 29, no. 7) Give an example of a vector space $V$ with endomorphisms $\theta$ and $\phi$ such that $V = \text{Im } \theta \oplus \text{Ker } \theta$, but $V \neq \text{Im } \phi \oplus \text{Ker } \phi$. 
10. Let $G$ be a group of order 15. Show that $G$ contains an element of order 3.

11. Let $H$ be a subgroup of a group $G$. Show that the map $a \mapsto a^{-1}$ determines a bijective map between the left cosets of $H$ and the right cosets of $H$.

12. Let $H$ be a subgroup of a group $G$ and let $g \in G$ be any element. Show that $g^{-1}Hg$ is a subgroup of $G$ which is isomorphic to $H$.

13. Determine whether or not the indicated map is a homomorphism: $\phi : \mathbb{R} \rightarrow GL(2, \mathbb{R})$ specified by $\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $\mathbb{R}$ is a group under $+$ and $GL(2, \mathbb{R})$ is the group of $2 \times 2$ invertible matrices with real entries, under multiplication of matrices.

14. Let $G := \mathbb{Z}/12\mathbb{Z}$ and write $\bar{n} = n + 12\mathbb{Z} \in G$ for each integer $n$. Find the order of the quotient group $G/\langle 8 \rangle$. Find the orders of the elements $2 + \langle 8 \rangle$ and $3 + \langle 8 \rangle$ in the quotient group $G/\langle 8 \rangle$.

15. (page 29, no. 5) (a) Let $U_1, U_2$ and $U_3$ be subspaces of a vector space $V$, with $V = U_1 + U_2 + U_3$. Show that

$$V = U_1 \oplus U_2 \oplus U_3 \iff U_1 \cap (U_2 + U_3) = U_2 \cap (U_3 + U_1) = U_3 \cap (U_1 + U_2) = \{0\}.$$ 

(b) Give an example of a vector space $V$ with three subspaces $U_1, U_2$ and $U_3$ such that $V = U_1 + U_2 + U_3$ and $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$, but $V \neq U_1 \oplus U_2 \oplus U_3$.

16. (page 29, no. 9) Suppose that $\theta$ is an endomorphism of the vector space $V$ and $\theta^2 = 1_V$. Show that $V = U \oplus W$, where $U = \{v \in V : v\theta = v\}$, $W = \{v \in V : v\theta = -v\}$. Deduce that $V$ has a basis $B$ such that $[\theta]_B$ is diagonal, with all diagonal entries equal to $+1$ or $-1$.

17. (page 29, no. 8) Let $V$ be a vector space and let $\theta$ be an endomorphism of $V$. Show that $\theta$ is a projection if and only if there is a basis $B$ of $V$ such that $[\theta]_B$ is diagonal, with all diagonal entries equal to $1$ or $0$. 

Some extra questions - do not hand in!

10. Let $G$ be a group of order 15. Show that $G$ contains an element of order 3.

11. Let $H$ be a subgroup of a group $G$. Show that the map $a \mapsto a^{-1}$ determines a bijective map between the left cosets of $H$ and the right cosets of $H$.

12. Let $H$ be a subgroup of a group $G$ and let $g \in G$ be any element. Show that $g^{-1}Hg$ is a subgroup of $G$ which is isomorphic to $H$.

13. Determine whether or not the indicated map is a homomorphism: $\phi : \mathbb{R} \rightarrow GL(2, \mathbb{R})$ specified by $\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $\mathbb{R}$ is a group under $+$ and $GL(2, \mathbb{R})$ is the group of $2 \times 2$ invertible matrices with real entries, under multiplication of matrices.