Math 5594 Homework 2, due Monday September 25, 2006 PJW

1. Let $G := \langle x \rangle$ be cyclic of order 15. Find the order of the element x^6 in G

2. Find the remainder of 9^{1573} when divided by 11.

3. Let G be a group of order 36. If G has an element $a \in G$ such that $a^{12} \neq 1$ and $a^{18} \neq 1$, show that G is cyclic.

4. Show that the mapping $G \to G$ specified by $x \to x^{-1}$ is a group homomorphism if and only if G is abelian.

5. Let $C_7 = \langle x \rangle$, $C_6 = \langle y \rangle$, $C_2 = \langle z \rangle$ be cyclic groups generated by elements x, y, z of orders 7, 6 and 2 respectively.
   (a) Is $\phi : C_6 \to C_2$ given by $\phi(y) = z$ a homomorphism?
   (b) Is $\phi : C_7 \to C_2$ given by $\phi(x) = z$ a homomorphism?

6. (a) Find all possible homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$.
   (b) Find all possible homomorphisms from $\mathbb{Z}$ onto $\mathbb{Z}$.

7. The quaternion group of order 8 is the set

$$Q_8 = \{1, -1, i, -i, j, -j, k, -k\}$$

with multiplication given by the rules

$$i^2 = j^2 = k^2 = -1,$$

$$ij = k, \quad jk = i, \quad ki = j, \quad ji = -k, \quad kj = -i, \quad ik = -j,$$

$$(-1)x = x(-1) = -x, \quad 1x = x1 = x$$

for all x (with the understanding that $-(-x) = x$). Write out a list of the orders of the elements of Q_8. Make a complete list of all the subgroups of Q_8 and determine which of them are normal. Show that the factor group $Q_8/\{1, -1\}$ is isomorphic to $C_2 \times C_2$.

8. Let $G := C_{15} = \langle x \rangle$ be cyclic of order 15. Find the order of the factor group $Q := G/\langle x^6 \rangle$. Find the order of the element $x^3\langle x^6 \rangle$ in Q. Find the order of the element $x^2\langle x^6 \rangle$ in Q.

9. (page 29, no. 7) Give an example of a vector space V with endomorphisms θ and ϕ such that $V = \operatorname{Im}\theta \oplus \operatorname{Ker}\theta$, but $V \neq \operatorname{Im}\phi \oplus \operatorname{Ker}\phi$.

Some extra questions - do not hand in!

10. Let G be a group of order 15. Show that G contains an element of order 3.

11. Let H be a subgroup of a group G. Show that the map $a \mapsto a^{-1}$ determines a bijective map between the left cosets of H and the right cosets of H.

12. Let H be a subgroup of a group G and let $g \in G$ be any element. Show that $g^{-1}Hg$ is a subgroup of G which is isomorphic to H.

13. Determine whether or not the indicated map is a homomorphism: $\phi : \mathbb{R} \to GL(2, \mathbb{R})$ specified by $\phi(x) = \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix}$ where $\mathbb{R}$ is a group under $+$ and $GL(2, \mathbb{R})$ is the group of 2×2 invertible matrices with real entries, under multiplication of matrices.

14. Let $G := \mathbb{Z}/12\mathbb{Z}$ and write $\bar{n} = n + 12\mathbb{Z} \in G$ for each integer n. Find the order of the quotient group $G/\langle \bar{8} \rangle$. Find the orders of the elements $\bar{2} + \langle \bar{8} \rangle$ and $\bar{3} + \langle \bar{8} \rangle$ in the quotient group $G/\langle \bar{8} \rangle$

15. (page 29, no. 5) (a) Let U_1, U_2 and U_3 be subspaces of a vector space V, with $V = U_1 + U_2 + U_3$. Show that

$$V = U_1 \oplus U_2 \oplus U_3 \Leftrightarrow U_1 \cap (U_2 + U_3) = U_2 \cap (U_3 + U_1) = U_3 \cap (U_1 + U_2) = \{0\}.$$

(b)Give an example of a vector space V with three subspaces U_1, U_2 and U_3 such that $V = U_1 + U_2 + U_3$ and $U_1 \cap U_2 = U_1 \cap U_3 = U_2 \cap U_3 = \{0\}$, but $V \neq U_1 \oplus U_2 \oplus U_3$.

16. (page 29, no. 9) Suppose that θ is an endomorphism of the vector space V and $\theta^2 = 1_V$. Show that $V = U \oplus W$, where $U = \{v \in V : v\theta = v\}$, $W = \{v \in V : v\theta = -v\}$. Deduce that V has a basis $\mathcal{B}$ such that $[\theta]_{\mathcal{B}}$ is diagonal, with all diagonal entries equal to $+1$ or -1.

17. (page 29, no. 8) Let V be a vector space and let θ be an endomorphism of V. Show that θ is a projection if and only if there is a basis $\mathcal{B}$ of V such that $[\theta]_{\mathcal{B}}$ is diagonal, with all diagonal entries equal to 1 or 0.