Math 8202 **Final Homework** **PJW**

Date due: 11am Monday May 18, 2009

There are seven questions altogether. Give careful and complete arguments. You may quote without proof any results which appear in the book by Rotman, any results from homework questions which you have been assigned, and any results which were lectured in class, provided that you indicate that you are doing this and that **they do not invalidate the question** (in my opinion). No other results may be quoted. Relevant arguments found in any book may be used, but should not be copied word for word. The work should be your own – please do not consult other people. I will be available to give clarification of the meaning of questions, but I will not give anyone hints. You may find it convenient to contact me by email: webb@math.umn.edu; or my office telephone: (612) 625 3491; or my home telephone: (507) 645 8150.

1. (Spring 2002, qn. 4) (14%) Let R be a commutative Noetherian ring with a 1 and I a maximal ideal of R.
   (a) (7%) Show that if M is a finitely generated R-module then M/IM has finite composition length as an R-module.
   (b) (7%) Show that R/I^{10} has finite composition length as an R-module.

2. (14%) Show that $\mathbb{Q}(\sqrt{3+\sqrt{5}})$ is the splitting field of a separable polynomial of degree 4. Determine the isomorphism type of the Galois group of $\mathbb{Q}(\sqrt{3+\sqrt{5}})$ over $\mathbb{Q}$.

3. (15%) Determine the Galois group of $(x^3-2)(x^3-3)$ over $\mathbb{Q}$. Determine all the subfields which contain $\mathbb{Q}(\rho)$ where ρ is a primitive 3rd root of unity.

4. (Spring 1994, no. 6) (14%)
   Let A be a real $m \times n$ matrix. Regarding A as the matrix of a linear map $\mathbb{R}^n \to \mathbb{R}^m$, let V be a subspace of $\mathbb{R}^n$ so that $\mathbb{R}^n = V \oplus \operatorname{Ker} A$. Show that the bilinear form $\langle\ ,\ \rangle$ defined on V by
   $$\langle u, v\rangle = u^T A^T A v \quad \text{for all } u, v \in V$$
   is non-singular, where ^{T} denotes the transpose. Hence show that the matrices A and $A^T A$ have the same rank.

PLEASE TURN OVER.

5. (Fall 2001, qn. 8) (14%) (a) (10) Let $(\ ,\ )$ be the bilinear form on $\mathbb{R}^3$ specified on column vectors $\mathbf{u}$ and $\mathbf{v}$ by $(\mathbf{u}, \mathbf{v}) = \mathbf{u}^T A \mathbf{v}$ where

$$A = \begin{pmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{pmatrix}$$

and $\mathbf{u}^T$ is the transpose of the vector $\mathbf{u}$. Determine whether or not this bilinear form is positive definite.
(b) (6) Give an example of a symmetric bilinear form on a real vector space which is not positive definite, and such that there is a basis $e_1, \ldots, e_n$ for the space with $(e_i, e_i) > 0$ for every i.

6. (Fall 2001) (14%) (a) (10) Let A be a finitely generated abelian group with a subgroup B with the property that whenever $na \in B$ for some $n \in \mathbb{Z}$ and $a \in A$ then $a \in B$. Show that $A \cong B \oplus A/B$.
[Additive notation is being used for these groups, so that na means $a + a + \cdots + a$ added n times. You may assume the structure theorem for finitely generated abelian groups.]
(b) (6) Let D be the subgroup of the free abelian group $C = \mathbb{Z}^3$ generated by the vector $(10, 6, 14)$. Show that C is not isomorphic to $D \oplus (C/D)$.

7. (Spring 2001) (15%) Let R be a commutative ring, $L = R^n$ a free R-module of rank n, and $A \in M_n(R)$ an $n \times n$ matrix viewed as an endomorphism of L.
(a) (5) Show that $\det(A) \cdot L \subseteq \mathrm{Im}(A)$.
(b) (9) If $R = \mathbb{Z}$ and $\det(A) \neq 0$, show that the size of $\mathrm{Coker}(A)$ equals $|\det(A)|$.