A (Review from last semester) (a) Let \( p(x) = x^3 + 9x + 6 \in \mathbb{Q}[x] \) and let \( \theta \) be a root of \( p \) in some extension field. Express \( (1 + \theta)^{-1} \) as a polynomial in \( \theta \).

(b) Same question with \( p(x) = x^3 - 2x - 2 \).

B Prove directly that the map \( a + b\sqrt{2} \mapsto a - b\sqrt{2} \) is an isomorphism of \( \mathbb{Q}(\sqrt{2}) \) with itself.

C Determine the degree over \( \mathbb{Q} \) of \( 2 + \sqrt{3} \) and of \( 1 + 3\sqrt{2} + 3\sqrt{4} \).

D* Prove that \( \mathbb{Q}(\sqrt{3} + \sqrt{5}) = \mathbb{Q}(\sqrt{3}, \sqrt{5}) \). Conclude that \( [\mathbb{Q}(\sqrt{3} + \sqrt{5}) : \mathbb{Q}] = 4 \). Find an irreducible polynomial satisfied by \( \sqrt{3} + \sqrt{5} \), giving justification that your polynomial is irreducible.

E Let \( F \) be a field of characteristic \( \neq 2 \). Let \( D_1 \) and \( D_2 \) be elements of \( F \), neither of which is a square in \( F \). Prove that \( F(\sqrt{D_1}, \sqrt{D_2}) \) is of degree 4 over \( F \) if \( D_1D_2 \) is not a square in \( F \) and is of degree 2 over \( F \) otherwise. When \( F(\sqrt{D_1}, \sqrt{D_2}) \) is of degree 4 over \( F \) the field is called a biquadratic extension of \( F \).

F Let \( F \) be a field of characteristic \( \neq 2 \). Let \( a, b \) be elements of the field \( F \) with \( b \) not a square in \( F \). Prove that a necessary and sufficient condition for \( \sqrt{a} + \sqrt{b} = \sqrt{m} + \sqrt{n} \) for some \( m \) and \( n \) in \( F \) is that \( a^2 - b \) is a square in \( F \). Use this to determine when the field \( \mathbb{Q}(\sqrt{a} + \sqrt{b}) \) with \( a, b \in \mathbb{Q} \) is biquadratic over \( \mathbb{Q} \).

G* Let \( K \) be an extension of \( F \) of degree \( n \).

(a) For any \( \alpha \in K \) prove that \( \alpha \) acting by left multiplication of \( K \) is an \( F \)-linear transformation of \( K \).

(b) Prove that \( K \) is isomorphic to a subfield of the ring of \( n \times n \) matrices of \( F \), so that ring of \( n \times n \) matrices over \( F \) contains an isomorphic copy of every extension of \( F \) of degree \( \leq n \).

H Let \( K = \mathbb{Q}(\sqrt{D}) \) for some squarefree integer \( D \). Let \( \alpha = a + b\sqrt{D} \) be an element of \( K \). Use the basis 1, \( \sqrt{D} \) for \( K \) as a vector space over \( \mathbb{Q} \) and show that the matrix of the linear transformation ‘multiplication by \( \alpha \)’ on \( K \) considered in the last exercise has the matrix \( \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \). Prove directly that the map \( a + b\sqrt{D} \mapsto \begin{pmatrix} a & bD \\ b & a \end{pmatrix} \) is an isomorphism of the field \( K \) with a subfield of the ring of \( 2 \times 2 \) matrices with coefficients in \( \mathbb{Q} \).

I* Determine the splitting field over \( \mathbb{Q} \), together with its degree over \( \mathbb{Q} \) for each of (a) \( x^4 - 2 \), (b) \( x^4 + 2 \), (c) \( x^4 + x^2 + 1 \) and (d) \( x^6 - 4 \).

J* (Fall 2000, qn. 5)(12%) Let \( K \supseteq k \) be a field extension and \( f \in k[X] \) an irreducible polynomial of degree relatively prime to the degree of the field extension \([K : k] \). Show that \( f \) is irreducible in \( K[X] \).