Math 8202 **Homework 10** **PJW**
Date due: April 20, 2009.

We will hold Quiz 5 April 27 and Quiz 6 on May 4. This is because the Algebra preliminary exam is on April 20.

Hand in only the 5 starred questions.

Section 8.2 page 548 no. 8.26

The only things we need from Section 8.2 are the chain conditions on modules (plus a lemma that the class of noetherian modules is closed under taking submodules, factor modules and extensions) and the Jordan-Hölder theorem 8.18 for modules, which requires the notion of a composition series. The proof I choose to give of this is exactly the same as the one which I gave for groups in the first semester. The Zassenhaus lemma 8.14 is an excellent result which has applications beyond the Jordan-Hölder theorem, but I do not think we need to do it here. I have never seen a question on the written exam which requires it.

After Section 8.2 we are going straight to Chapter 9, and we are going to do the structure theorem for finitely generated modules over a PID (Section 9.1) but in the special case of a Euclidean domain using the theory of Smith normal form. This is described Section 9.4, so the first thing we will do in Chapter 9 is Section 9.4. I hope to get all this done this week.

You can do questions EE, FF and GG right now, without any of the theory to be described this week.

BB*. (Fall 2000, qn. 3) Let S be the ring $\mathbb{R}[X]/(X^2+1)^3$. Prove that the free S-module $S \oplus S$ has a composition series. Find the length of a composition series of $S \oplus S$.

CC*. (Spring 2001, qn. 5) Let M be a module over a ring R which has a composition series

$$0 = M_0 \subset M_1 \subset M_2 \subset \cdots \subset M_{n-1} \subset M_n = M.$$

(a) Suppose that the modules in this composition series form a complete list of all the submodules of M. Show that it is not possible to write M as a direct sum of submodules $M = N_1 \oplus N_2$ unless either $N_1 = 0$ or $N_2 = 0$. Show further that every submodule of M can be generated by a single element.

(b) Show by example that it is possible to have a module M which simultaneously has the properties

- every submodule of M can be generated by a single element,
- M has a composition series, and
- there are submodules of M other than the ones in the composition series.

DD. (Spring 2002, qn. 4) (12%) Let R be a commutative Noetherian ring with a 1 and I a maximal ideal of R.

(a) (6%) Show that if M is a finitely generated R-module then M/IM has finite composition length as an R-module.
(b) (6%) Show that R/I^{10} has finite composition length as an R-module.

In the following questions, the *rank* of any module M over an integral domain R is defined as the largest number of R-linearly independent elements of M.

EE Let R be an integral domain.
(a) Show that the rank of R^n is n.
(b) Show that if N is a submodule the R-module M then $\operatorname{rank} N \leq \operatorname{rank} M$.

FF* Let M be a module over the integral domain R.
(a) Suppose that M has rank n and that $x_1, \ldots, x_n$ is any maximal set of linearly independent elements of M. Let $N = Rx_1 + \cdots + Rx_n$ be the submodule generated by $x_1, x_2, \ldots, x_n$. Prove that N is isomorphic to R^n and that the quotient M/N is a torsion R-module. [A torsion module is one in which every element has non-zero annihilator.]
(b) Prove conversely that if M contains a submodule N that is free of rank n (i.e. $N \cong R^n$) such that the quotient M/N is a a torsion R-module then M has rank n. [Let $y_1, y_2, \ldots, y_{n+1}$ be any $n+1$ elements of M Use the fact that M/N is torsion to write $r_i y_i$ as a linear combination of a basis for N for some nonzero elements $r_1, \ldots, r_{n+1}$ of R. Argue that the $r_i y_i$ and hence also the y_i, are linearly dependent.]

GG Let R be an integral domain, let M be an R-module and let N be a submodule of M. Suppose M has rank n, N has rank r and the quotient M/N as rank s. Prove that $n = r + s$. [Let $x_1, x_2, \ldots, x_s$ be elements of M whose images in M/N are a maximal set of independent elements and let $x_{s+1}, \ldots, x_{s+r}$ be a maximal set of independent elements in N. Prove that $x_1, x_2, \ldots, x_{s+r}$ are linearly independent in M and that for any element $Y \in M$ there is a nonzero element $r \in R$ such that ry is a linear combination of these elements. Then use exercise EE.]

HH* (Modification of Fall 1993, qn. 8) Let M be the subgroup of $\mathbb{Z}^3$ generated by the three vectors $(2,4,4)$, $(6,3,-6)$ and $(4,14,20)$.
(a) Calculate the rank of M.
(b) Calculate the invariant factors and the elementary divisors of $\mathbb{Z}^3/M$.
(c) Find a basis f_1, f_2, f_3 for $\mathbb{Z}^3$ with the property that $a_1 f_1, \ldots, a_r f_r$ is a basis for M, where r is the rank of M, and where $a_1 \mid \cdots \mid a_r$.

II* Let $A = \mathbb{Z}/12\mathbb{Z} \oplus \mathbb{Z}/15\mathbb{Z} \oplus \mathbb{Z}/18\mathbb{Z} \oplus \mathbb{Z}/27\mathbb{Z}$.
(a) Calculate the invariant factors of A.
(b) Calculate the elementary divisors of A.
(c) Calculate the structure of the group $3A/9A$.

The following is a collection of past exam questions that are relevant for the material we are now covering. Some of them use ideas (notably the idea of a projective module) which we have not yet done. These questions are included here only for your information – you are not asked to do any of them!

JJ. (Spring 1999) (a) (9 pts) Let A be an $n \times n$ matrix with integer entries. Regarding the free abelian group $\mathbb{Z}^n$ as the set of column vectors of length n with integer entries, let H be the subgroup of $\mathbb{Z}^n$ generated by the columns of A. Prove that the group $\mathbb{Z}^n/H$ is finite if and only if $\det A \neq 0$.
(b) (5 pts) Give an example of two subgroups of the group $\mathbb{Z} \oplus \mathbb{Z}$ each of which is a direct summand of $\mathbb{Z} \oplus \mathbb{Z}$ but such that their sum is not a direct summand of $\mathbb{Z} \oplus \mathbb{Z}$. Give reasons for your assertions.

KK. (Spring 2001) (14%) Let R be a commutative ring, $L = R^n$ a free R-module of rank n, and $A \in M_n(R)$ an $n \times n$ matrix viewed as an endomorphism of L.
(a) (5) Show that $\det(A) \cdot L \subseteq \mathrm{Im}(A)$.
(b) (9) If $R = \mathbb{Z}$ and $\det(A) \neq 0$, show that the size of $\mathrm{Coker}(A)$ equals $|\det(A)|$.

LL. (Fall 2001) (11%) (a) (7) Let A be a finitely generated abelian group with a subgroup B with the property that whenever $na \in B$ for some $n \in \mathbb{Z}$ and $a \in A$ then $a \in B$. Show that $A \cong B \oplus A/B$.
[Additive notation is being used for these groups, so that na means $a + a + \cdots + a$ added n times. You may assume the structure theorem for finitely generated abelian groups.]
(b) (4) Let D be the subgroup of the free abelian group $C = \mathbb{Z}^3$ generated by the vector $(10, 6, 14)$. Show that C is not isomorphic to $D \oplus (C/D)$.

MM. (Spring 2002) (15%) Let A be a finitely generated abelian group, let B be a subgroup and put $C = A/B$. Suppose that

$$A = \mathbb{Z}^u \oplus F_A,$$
$$B = \mathbb{Z}^v \oplus F_B,$$
$$C = \mathbb{Z}^w \oplus F_C,$$

where F_A, F_B and F_C are finite abelian groups.
(a) (9%) Show that $u = v + w$.
[If you use properties of the tensor product, they should be proved. You may assume the Structure Theorem for finitely generated abelian groups.]
(b) (6%) Suppose further that $F_C = 0$. Show that $F_B = F_A$.

NN. (Fall 2002) (14%) Let $A = \mathbb{Z}^3$ be a free abelian group of rank 3, and let B be the subgroup of A generated by the elements $(2, -4, -1)$, $(4, 1, 1)$ and $(-2, -2, 1)$ (where we regard elements of A as row vectors of length 3 with integer entries). Writing

$$A/B = \mathbb{Z}^t \oplus \mathbb{Z}/d_1\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/d_s\mathbb{Z},$$

calculate the values of the integers $t, d_1, \ldots, d_s$.