Math 8202  Homework 5  PJW

Date due: March 2, 2009. Hand in only the 5 starred questions (all of the questions - I apologize for not having a greater selection of questions this week! :-) ).

DDD* Suppose \( f(x) \in \mathbb{Z}[x] \) is an irreducible quartic whose splitting field has Galois group \( S_4 \) over \( \mathbb{Q} \) (there are many such quartics). Let \( \theta \) be a root of \( f(x) \) and set \( K = \mathbb{Q}(\theta) \). Prove that \( K \) is an extension of \( \mathbb{Q} \) of degree 4 with no proper subfields. Are there any Galois extensions of \( \mathbb{Q} \) of degree 4 with no proper subfields?

EEE* (Fall 2002, qn. 6) Let \( a \) be a nonzero rational number.
   (a) (6%) Determine the values of \( a \) such that the extension \( \mathbb{Q}(\sqrt{ai}) \) is of degree 4 over \( \mathbb{Q} \), where \( i^2 = -1 \).
   (b) (12%) When \( K = \mathbb{Q}(\sqrt{ai}) \) is of degree 4 over \( \mathbb{Q} \) show that \( K \) is Galois over \( \mathbb{Q} \) with the Klein 4-group as Galois group. In this case determine all the quadratic extensions of \( \mathbb{Q} \) contained in \( K \).

FFF* On page 211 of Rotman’s book, for a finite extension \( K = k(\alpha_1, \ldots, \alpha_n) \) of a field \( k \), a normal closure of \( K/k \) is defined to be an extension \( E \supseteq K \) of least degree which is the splitting field of some polynomial \( f \in k[x] \). No assertion of the uniqueness of \( E \) is made. Show that if each of the minimal polynomials of the \( \alpha_i \) over \( k \) is separable, then the normal closure of \( K/k \) is unique.

GGG* Let \( L \) be the normal closure of a finite extension \( \mathbb{Q}(\alpha) \) of \( \mathbb{Q} \). For any prime \( p \) dividing the order of Gal(\( L/\mathbb{Q} \)) prove that there is a subfield \( F \) of \( \mathbb{O} \) with \([L:F] = p \) and \( L = F(\alpha) \).

HHH* Let \( F \) be a subfield of the real numbers \( \mathbb{R} \). Let \( a \) be an element of \( F \) and let \( K = F(\sqrt[n]{a}) \) where \( \sqrt[n]{a} \) denotes a real \( n \)th root of \( a \). Prove that if \( L \) is any Galois extension of \( F \) contained in \( K \) then \([L:F] \leq 2 \).