Math 3593 Practice for the final exam.

You will **not** be allowed to use books, notes or a calculator on this exam. At the top of the exam you will be given the following formulas, as well as Taylor expansions of standard functions if you need them, but I don't think you do:

Formulas

Polar coordinates: $x = r\cos\theta$, $y = r\sin\theta$; $dx\,dy = r\,dr\,d\theta$.
Cylindrical coordinates: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$; $dx\,dy\,dz = r\,dr\,d\theta\,dz$.
Spherical polars: $x = r\cos\phi\cos\theta$, $y = r\cos\phi\sin\theta$, $z = r\sin\phi$; $dx\,dy\,dz = r^2\cos\phi\,dr\,d\phi\,d\theta$.

1. Prove that a subset of a set of volume zero has volume zero.

2. Find the surface area of the part of the graph of the function $z = y^2 - x^2$ which lies above the circle $x^2 + y^2 \leq 1$ in the xy-plane.

3. Let A be the unit circle $x^2 + y^2 \leq 1$ and let $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ be the transformation given by $\Phi\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 2x + y \\ x - y \end{pmatrix}$. Find the area of $\Phi(A)$.

5. Find the length of the part of the helical spiral in $\mathbb{R}^3$, specified in cylindrical polar coordinates (r, θ, z) by $r = 2$, $z = 3\theta$, for which $0 \leq \theta \leq 2\pi$.

6. Find the area of the plane elliptical region which is the part of the plane $z = 4 - x - 2y$ that lies above the circle $x^2 + y^2 \leq 1$ in the xy-plane.

7. Let ϕ be the angle between a vector in $\mathbb{R}^3$ and the z-axis. Find the volume of the region in $\mathbb{R}^3$ bounded by the surface given in spherical polar coordinates by $r = 3(1 - \cos\phi)$.

8. THIS ONE APPEARED ON HW: Let $S = \partial B$ be the closed surface that is the boundary of the hemisphere

$$B: \qquad x^2 + y^2 + z^2 \leq 1, \quad z \geq 0.$$

Thus S is the union of the flat unit disc S_1 in the xy-plane given as

$$S_1: \qquad x^2 + y^2 \leq 1, \quad z = 0$$

and the curved surface S_2 given as

$$S_2: \qquad x^2 + y^2 + z^2 = 1, \quad z \geq 0.$$

Suppose that S is oriented with normal vector pointing out from the hemisphere at each point, and let S_1 and S_2 have this same orientation. Let F be the vector field $F(x, y, z) = (x + \cos y + \cos z, y + \sqrt{x^2 + 1}\ln(z^2 + 1), z + 3)$.
(a) (4) Compute $\int_S F \cdot dS$.
(b) (4) Compute $\int_{S_1} F \cdot dS$.
(c) (4) Compute $\int_{S_2} F \cdot dS$.

9. (20%) Calculate

$$\int_\gamma (y - \tan^{-1}\sqrt{x + 10})dx + (3x + e^{y^2}\sin y)dy$$

where γ is the boundary of the region enclosed by the parabola $y = x^2$ and the line $y = 4$.

10. Let C be the curve in $\mathbb{R}^2$ parametrized by $\gamma(t) = \begin{pmatrix} t - t^2 \\ t - t^3 \end{pmatrix}$ where $0 \leq t \leq 1$, taken with the orientation given by this parametrization. You may assume that this curve is a loop which does not cross itself, and that it is in fact the boundary of a 2-manifold with boundary, namely the region enclosed by C.
   (a) Calculate $\int_C y\, dx$.
   (b) By expressing the integral in (a) as a double integral (using Green's theorem), calculate the area of the region enclosed by C.

11. For each of the following sets, determine whether or not it is a smooth manifold, justifying your conclusion.
   (1) The set of 2×2 real matrices A such that $A^2 = I$.
   (2) The set of points $\binom{x}{y}$ in $\mathbb{R}^2$ for which x and y have the same sign, or are both zero.
   (3) $\{x \in \mathbb{R} \mid x > 0\}$
   (4) $\mathbb{R} - \{0\}$
   (5) The union of the coordinate axes $x = 0$ and $y = 0$ in $\mathbb{R}^2$.

12. Find the maximum and minimum values of $f\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x^2 + xy + 2y^2 - z^2$ on the ball $x^2 + y^2 + z^2 \leq 100$.

Plus: the questions which have appeared on the previous practice handouts, and
page 275 Section 2.10: nos 2, 4, 7, 8, 14, 15, 16
page 278 Section 2.11: nos 2.29, 2.30, 2.31
page 367 Section 3.7: no 6 Take this function and find its maximum and minimum values on $x^2 + y^2 + z^2 \leq 10$. (I have not done this modified problem and am not sure if it will work out appropriately.)

p. 386: 3.1, 3.2, 3.5, 3.10, 3.20
p. 404: 4.1.10, 4.1.14
p. 445: 4.5.7, 4.5.8, 4.5.11, 4.5.12, 4.5.14, 4.5.15, 4.5.16, 4.5.18
p. 474: 4.8.1, 4.8.2, 4.8.4, 4.8.13
p. 493: 4.10.8, 4.10.12, 4.10.13, 4.10.14, 4.10.17, 4.10.18, 4.10.19
p. 514: 4.11, 4.12, 4.13, 4.21, 4.23
p. 522: 5.1.1, 5.1.2
p. 540: 5.3.2, 5.3.6, 5.3.8, 5.3.9, 5.3.15, 5.3.18, 5.3.21
p. 547: 5.3, 5.4