As practice, but not part of the homework, make sure you can do questions in Rotman apart from the ones listed below, such as 2.19a.

Assignment questions:
Rotman pages 64-69: 2.13, 2.18, 2.20 (assume without proof Proposition 2.42).
Rotman pages 94-97: 2.28, 2.34, 2.36(i)
Questions 1, 2, 3 below.

0. Preliminary facts for questions 1 and 2, to be thought about, but not written down or handed in.
   (a) A small category with at most one morphism between any two objects is the same thing as a preordered set, namely, a set with a transitive binary operation.
   (b) For any small category $C$ we may define a new category $C_1$ with the same objects and where there is a unique homomorphism $x \to y$ in $C_1$ if and only if $\text{Hom}_C(x, y) \neq \emptyset$. Then $C_1$ is a preordered set, and there is a functor $F_1: C \to C_1$ with the property that whenever $G: C \to D$ is a functor where $D$ is a preordered set, then $G$ can be factored $G = H \circ F_1$ for some unique functor $H: C_1 \to D$.

1. For any small category $C$, show that the following is an equivalence relation on the objects: $x \sim y \iff$ there are morphisms $x \to y$ and $y \to x$. Writing $x$ for the equivalence class of $x$, show that we may define a category $C_2$ with these equivalence classes as objects, and where there is a morphism unique morphism $\underline{x} \to y$ in $C_2$ if and only if there is a morphism $x \to y$ in $C$. Show that $C_2$ is a poset, and that there is a functor $F_2: C \to C_2$ with the property that whenever $G_2: C \to \mathcal{P}$ is a functor where $\mathcal{P}$ is a poset then $G_2$ can be factored $G_2 = H \circ F_2$ for some unique functor $H: C_2 \to \mathcal{P}$.

2. Let $C$ be a category which is a preordered set. Show that $C$ is equivalent to a poset.

3. (A more specific version of 2.27) Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ be the matrix of $S: V \to V$ and let $B = \begin{pmatrix} 1 & 0 \\ 1 & 2 \end{pmatrix}$ be the matrix of $T: W \to W$, where $V$ is a vector space with basis $\{v_1, v_2\}$ and $W$ is a vector space with basis $\{w_1, w_2\}$. Assuming without proof that $\{v_i \otimes w_j \mid i, j \in \{1, 2\}\}$ is a basis for $V \otimes W$, put this set in the correct order for the matrix of $\frac{\delta}{\delta} \otimes T$ to be

$$
\begin{pmatrix}
1 & 0 & 2 & 0 \\
2 & 2 & 4 \\
3 & 0 & 4 & 0 \\
3 & 6 & 4 & 8 \\
\end{pmatrix}
$$
Proof.

(i) \( p \) is surjective.
Let \( M = B''/\text{im} \, p \) and let \( f : B'' \rightarrow B''/\text{im} \, p \) be the natural map, so that \( f \in \text{Hom}(B'', M) \). Then \( p^*(f) = fp = 0 \), so that \( f = 0 \), because \( p^* \) is injective. Therefore, \( B''/\text{im} \, p = 0 \), and \( p \) is surjective.

(ii) \( \text{im} \, i \subseteq \ker \, p \).
Since \( i^* \, p^* = 0 \), we have \( 0 = (pi)^* \). Hence, if \( M = B'' \) and \( g = 1_{B''} \), so that \( g \in \text{Hom}(B'', M) \), then \( 0 = (pi)^*g = gpi = pi \), and so \( \text{im} \, i \subseteq \ker \, p \).

(iii) \( \ker \, p \subseteq \text{im} \, i \).
Now choose \( M = B/\text{im} \, i \) and let \( h : B \rightarrow M \) be the natural map, so that \( h \in \text{Hom}(B, M) \). Clearly, \( i^*h = hi = 0 \), so that exactness of the Hom sequence gives an element \( h' \in \text{Hom}_R(B'', M) \) with \( p^*(h') = h' \, p = h \). We have \( \text{im} \, i \subseteq \ker \, p \), by part (ii); hence, if \( \text{im} \, i \neq \ker \, p \), there is an element \( b \in B \) with \( b \notin \text{im} \, i \) and \( b \in \ker \, p \). Thus, \( hb \neq 0 \) and \( pb = 0 \), which gives the contradiction \( hb = h' \, pb = 0 \).

The single condition that \( i^* : \text{Hom}_R(B, M) \rightarrow \text{Hom}_R(B', M) \) be surjective is much stronger than the hypotheses of Proposition 2.42 (see Exercise 2.20 on page 68).

Exercises

Unless we say otherwise, all modules in these exercises are left \( R \)-modules.

2.1 Let \( R \) and \( S \) be rings, and let \( \varphi : R \rightarrow S \) be a ring homomorphism. If \( M \) is a left \( S \)-module, prove that \( M \) is also a left \( R \)-module if we define
\[
rm = \varphi(r)m,
\]
for all \( r \in R \) and \( m \in M \).

2.2 Give an example of a left \( R \)-module \( M = S \oplus T \) having a submodule \( N \) such that \( N \neq (N \cap S) \oplus (N \cap T) \).

*2.3 Let \( f, g : M \rightarrow N \) be \( R \)-maps between left \( R \)-modules. If \( M = \langle X \rangle \) and \( f|X = g|X \), prove that \( f = g \).

*2.4 Let \( (M_i)_{i \in I} \) be a (possibly infinite) family of left \( R \)-modules and, for each \( i \), let \( N_i \) be a submodule of \( M_i \). Prove that
\[
\left( \bigoplus_i M_i \right) / \left( \bigoplus_i N_i \right) \cong \bigoplus_i (M_i / N_i).
\]
2.1 Modules

*2.5* Let 0 → A → B → C → 0 be a short exact sequence of left $R$-modules. If $M$ is any left $R$-module, prove that there are exact sequences

$$0 \to A \oplus M \to B \oplus M \to C \to 0$$

and

$$0 \to A \to B \oplus M \to C \oplus M \to 0.$$

*2.6* (i) Let $A_{n+1} \xrightarrow{d_{n+1}} A_n \xrightarrow{d_n} A_{n-1}$ be an exact sequence, and let $\text{im} d_{n+1} = K_n = \ker d_n$ for all $n$. Prove that

$$0 \to K_n \xrightarrow{i_n} A_n \xrightarrow{d'_n} K_{n-1} \to 0$$

is an exact sequence for all $n$, where $i_n$ is the inclusion and $d'_n$ is obtained from $d_n$ by changing its target. We say that the original sequence has been factored into these short exact sequences.

(ii) Let

$$A_1 \xrightarrow{f_1} A_0 \xrightarrow{f_0} K \to 0$$

and

$$0 \to K \xrightarrow{g_0} B_0 \xrightarrow{g_1} B_1 \to 0$$

be exact sequences. Prove that

$$A_1 \xrightarrow{f_1} A_0 \xrightarrow{g_0} B_0 \xrightarrow{g_1} B_1 \to 0$$

is an exact sequence. We say that the original two sequences have been spliced to form the new exact sequence.

*2.7* Use left exactness of Hom to prove that if $G$ is an abelian group, then $\text{Hom}_\mathbb{Z}(I_n, G) \cong G[n]$, where $G[n] = \{ g \in G : ng = 0 \}$.

*2.8* (i) Prove that a short exact sequence in $\mathbb{R} \text{Mod}$,

$$0 \to A \xrightarrow{i} B \xrightarrow{p} C \to 0,$$

splits if and only if there exists $q : B \to A$ with $qi = 1_A$. (Note that $q$ is a retraction $B \to \text{im} i$.)

(ii) A sequence $A \xrightarrow{i} B \xrightarrow{p} C$ in $\text{Groups}$ is exact if $\text{im} i = \ker p$; an exact sequence

$$1 \to A \xrightarrow{i} B \xrightarrow{p} C \to 1$$

in $\text{Groups}$ is split if there is a homomorphism $j : C \to B$ with $pj = 1_C$. Prove that $1 \to A_3 \xrightarrow{i} S_3 \xrightarrow{p} \mathbb{I}_2 \to 1$ is a split exact sequence. In contrast to part (i), show, in a split exact sequence in $\text{Groups}$, that there may not be a homomorphism $q : B \to A$ with $qi = 1_A$. 

2.9 (i) Let \( v_1, \ldots, v_n \) be a basis of a vector space \( V \) over a field \( k \). Let \( v_i^* : V \to k \) be the evaluation \( V^* \to k \) defined by \( v_i^* = (\square, v_i) \) (see Example 1.16). Prove that \( v_1^*, \ldots, v_n^* \) is a basis of \( V^* \) (it is called the dual basis of \( v_1, \ldots, v_n \)).

**Hint.** Use Corollary 2.22(ii) and Example 2.27.

(ii) Let \( f : V \to V \) be a linear transformation, and let \( A \) be the matrix of \( f \) with respect to a basis \( v_1, \ldots, v_n \) of \( V \); that is, the \( i \)th column of \( A \) consists of the coordinates of \( f(v_i) \) with respect to the given basis \( v_1, \ldots, v_n \). Prove that the matrix of the induced map \( f^* : V^* \to V^* \) with respect to the dual basis is the transpose \( A^T \) of \( A \).

2.10 If \( X \) is a subset of a left \( R \)-module \( M \), prove that \( \langle X \rangle \), the submodule of \( M \) generated by \( X \), is equal to \( \bigcap S \), where the intersection ranges over all those submodules \( S \) of \( M \) that contain \( X \).

2.11 Prove that if \( f : M \to N \) is an \( R \)-map and \( K \) is a submodule of a left \( R \)-module \( M \) with \( K \subseteq \ker f \), then \( f \) induces an \( R \)-map \( \widehat{f} : M/K \to N \) by \( \widehat{f} : m + K \mapsto f(m) \).

2.12 (i) Let \( R \) be a commutative ring and let \( J \) be an ideal in \( R \). Recall Example 2.8(iv): if \( M \) is an \( R \)-module, then \( JM \) is a submodule of \( M \). Prove that \( M/JM \) is an \( R/J \)-module if we define scalar multiplication:

\[
(r + J)(m + JM) = rm + JM.
\]

Conclude that if \( JM = \{0\} \), then \( M \) itself is an \( R/J \)-module. In particular, if \( J \) is a maximal ideal in \( R \) and \( JM = \{0\} \), then \( M \) is a vector space over \( R/J \).

(ii) Let \( I \) be a maximal ideal in a commutative ring \( R \). If \( X \) is a basis of a free \( R \)-module \( F \), prove that \( F/IF \) is a vector space over \( R/I \) and that \( \{\text{cosets } x + IF : x \in X \} \) is a basis.

2.13 Let \( M \) be a left \( R \)-module.

(i) Prove that the map \( \varphi_M : \text{Hom}_R(R, M) \to M \), given by \( \varphi_M : f \mapsto f(1) \), is an \( R \)-isomorphism.

**Hint.** Make the abelian group \( \text{Hom}_R(R, M) \) into a left \( R \)-module by defining \( rf \) (for \( f : R \to M \) and \( r \in R \)) by \( rf : s \mapsto f(sr) \) for all \( s \in R \).

(ii) If \( g : M \to N \), prove that the following diagram commutes:

\[
\begin{array}{ccc}
\text{Hom}_R(R, M) & \xrightarrow{\varphi_M} & M \\
\downarrow{g_*} & & \downarrow{g} \\
\text{Hom}_R(R, N) & \xrightarrow{\varphi_N} & N.
\end{array}
\]
2.1 Modules

Conclude that \( \varphi = (\varphi_M)_{M \in \text{obj}(\mathcal{R}\text{Mod})} \) is a natural isomorphism from Hom\(_R(R, \square)\) to the identity functor on \( \mathcal{R}\text{Mod} \). [Compare with Example 1.16(ii).]

2.14 Let \( A \overset{f}{\rightarrow} B \overset{g}{\rightarrow} C \) be a sequence of module maps. Prove that \( gf = 0 \) if and only if \( \text{im } f \subseteq \ker g \). Give an example of such a sequence that is not exact.

*2.15 (i)\) Prove that \( f : M \rightarrow N \) is surjective if and only if \( \text{coker } f = \{0\} \).

(ii) If \( f : M \rightarrow N \) is a map, prove that there is an exact sequence

\[
0 \rightarrow \ker f \rightarrow M \overset{f}{\rightarrow} N \rightarrow \text{coker } f \rightarrow 0.
\]

*2.16 (i)\) If \( 0 \rightarrow M \rightarrow 0 \) is an exact sequence, prove that \( M = \{0\} \).

(ii) If \( A \overset{f}{\rightarrow} B \overset{g}{\rightarrow} C \overset{h}{\rightarrow} D \) is an exact sequence, prove that \( f \) is surjective if and only if \( h \) is injective.

(iii) Let \( A \overset{\alpha}{\rightarrow} B \overset{\beta}{\rightarrow} C \overset{\gamma}{\rightarrow} D \overset{\delta}{\rightarrow} E \) be exact. If \( \alpha \) and \( \delta \) are isomorphisms, prove that \( C = \{0\} \).

*2.17 If \( A \overset{f}{\rightarrow} B \overset{g}{\rightarrow} C \overset{h}{\rightarrow} D \overset{k}{\rightarrow} E \) is exact, prove that there is an exact sequence

\[
0 \rightarrow \text{coker } f \overset{\alpha}{\rightarrow} C \overset{\beta}{\rightarrow} \ker k \rightarrow 0,
\]

where \( \alpha : b + \text{im } f \mapsto gb \) and \( \beta : c \mapsto hc \).

*2.18 Let \( 0 \rightarrow A \overset{i}{\rightarrow} B \overset{p}{\rightarrow} C \rightarrow 0 \) be a short exact sequence.

(i) Assume that \( A = \langle X \rangle \) and \( C = \langle Y \rangle \). For each \( y \in Y \), choose \( y' \in B \) with \( p(y') = y \). Prove that

\[
B = \{i(X) \cup \{y' : y \in Y\}\}.
\]

(ii) Prove that if both \( A \) and \( C \) are finitely generated, then \( B \) is finitely generated. More precisely, prove that if \( A \) can be generated by \( m \) elements and \( C \) can be generated by \( n \) elements, then \( B \) can be generated by \( m + n \) elements.

*2.19 Let \( R \) be a ring, let \( A \) and \( B \) be left \( R \)-modules, and let \( r \in Z(R) \).

(i) If \( \mu_r : B \rightarrow B \) is multiplication by \( r \), prove that the induced map \( (\mu_r)_* : \text{Hom}_R(A, B) \rightarrow \text{Hom}_R(A, B) \) is also multiplication by \( r \).

(ii) If \( m_r : A \rightarrow A \) is multiplication by \( r \), prove that the induced map \( (m_r)^* : \text{Hom}_R(A, B) \rightarrow \text{Hom}_R(A, B) \) is also multiplication by \( r \).
*2.20 Suppose one assumes, in the hypothesis of Proposition 2.42, that the induced map $i^*$: $\text{Hom}_R(B, M) \rightarrow \text{Hom}_R(B', M)$ is surjective for every $M$. Prove that $0 \rightarrow B' \xrightarrow{i} B \xrightarrow{p} B'' \rightarrow 0$ is a split short exact sequence.

*2.21 If $T: \text{Ab} \rightarrow \text{Ab}$ is an additive functor, prove, for every abelian group $G$, that the function $\text{End}(G) \rightarrow \text{End}(TG)$, given by $f \mapsto Tf$, is a ring homomorphism.

*2.22 (i) Prove that $\text{Hom}_\mathbb{Z}(\mathbb{Q}, C) = \{0\}$ for every cyclic group $C$.

(ii) Let $R$ be a commutative ring. If $M$ is an $R$-module such that $\text{Hom}_R(M, R/I) = \{0\}$ for every nonzero ideal $I$, prove that $\text{im} f \subseteq \bigcap I$ for every $R$-map $f: M \rightarrow R$, where the intersection is over all nonzero ideals $I$ in $R$.

(iii) Let $R$ be a domain and suppose that $M$ is an $R$-module with $\text{Hom}_R(M, R/I) = \{0\}$ for all nonzero ideals $I$ in $R$. Prove that $\text{Hom}_R(M, R) = \{0\}$.

Hint. Every $r \in \bigcap_{I \neq 0} I$ is nilpotent.

2.23 Generalize Proposition 2.26. Let $(S_i)_{i \in I}$ be a family of submodules of a left $R$-module $M$. If $M = \bigcup_{i \in I} S_i$, then the following conditions are equivalent.

(i) $M = \bigoplus_{i \in I} S_i$.

(ii) Every $a \in M$ has a unique expression of the form $a = s_{i_1} + \cdots + s_{i_n}$, where $s_{i_j} \in S_{i_j}$.

(iii) $S_i \cap \bigcup_{j \neq i} S_j = \{0\}$ for each $i \in I$.

*2.24 (i) Prove that any family of $R$-maps $(f_j: U_j \rightarrow V_j)_{j \in J}$ can be assembled into an $R$-map $\varphi: \bigoplus_{j \in J} U_j \rightarrow \bigoplus_{j \in J} V_j$, namely, $\varphi: (u_j) \mapsto (f_j(u_j))$.

(ii) Prove that $\varphi$ is an injection if and only if each $f_j$ is an injection.

*2.25 (i) If $Z_i \cong \mathbb{Z}$ for all $i$, prove that

$$\text{Hom}_\mathbb{Z}\left(\prod_{i=1}^{\infty} Z_i, \mathbb{Z}\right) \not\cong \prod_{i=1}^{\infty} \text{Hom}_\mathbb{Z}(Z_i, \mathbb{Z}).$$

Hint. A theorem of J. Łos and, independently, of E. C. Zeeman (see Fuchs, *Infinite Abelian Groups* II, Section 94) says that

$$\text{Hom}_\mathbb{Z}\left(\prod_{i=1}^{\infty} Z_i, \mathbb{Z}\right) \cong \bigoplus_{i=1}^{\infty} \text{Hom}_\mathbb{Z}(Z_i, \mathbb{Z}) \cong \bigoplus_{i=1}^{\infty} Z_i.$$
(i) Let $p$ be a prime and let $B_n$ be a cyclic group of order $p^n$, where $n$ is a positive integer. If $A = \bigoplus_{n=1}^{\infty} B_n$, prove that

$$\text{Hom}_k\left( A, \bigoplus_{n=1}^{\infty} B_n \right) \not\cong \bigoplus_{n=1}^{\infty} \text{Hom}_k(A, B_n).$$

**Hint.** Prove that $\text{Hom}(A, A)$ has an element of infinite order, while every element in $\bigoplus_{n=1}^{\infty} \text{Hom}_k(A, B_n)$ has finite order.

(ii) Prove that $\text{Hom}_\mathbb{Z}(\prod_{n \geq 2} \mathbb{Z}_n, \mathbb{Q}) \not\cong \prod_{n \geq 2} \text{Hom}_\mathbb{Z}(\mathbb{Z}_n, \mathbb{Q})$.

**2.2 Tensor Products**

One of the most compelling reasons to introduce tensor products comes from Algebraic Topology. The homology groups of a space are interesting (for example, computing the homology groups of spheres enables us to prove the Jordan Curve Theorem), and the homology groups of the cartesian product $X \times Y$ of two topological spaces are computed (by the Künneth formula) in terms of the tensor product of the homology groups of the factors $X$ and $Y$.

Here is a second important use of tensor products. We saw, in Example 2.2, that if $k$ is a field, then every $k$-representation $\varphi: H \to \text{Mat}_n(k)$ of a group $H$ to $n \times n$ matrices makes the vector space $k^n$ into a left $kH$-module;
Exercises

2.27 Let $V$ and $W$ be finite-dimensional vector spaces over a field $F$, say, and let $v_1, \ldots, v_m$ and $w_1, \ldots, w_n$ be bases of $V$ and $W$, respectively. Let $S: V \rightarrow V$ be a linear transformation having matrix $A = [a_{ij}]$, and let $T: W \rightarrow W$ be a linear transformation having matrix $B = [b_{k\ell}]$. Show that the matrix of $S \otimes T: V \otimes_k W \rightarrow V \otimes_k W$, with respect to a suitable listing of the vectors $v_i \otimes w_j$, is the $nm \times nm$ matrix $K$, which we write in block form:

$$A \otimes B = \begin{bmatrix}
a_{11}B & a_{12}B & \cdots & a_{1m}B \\
a_{21}B & a_{22}B & \cdots & a_{2m}B \\
& & & \\
a_{m1}B & a_{m2}B & \cdots & a_{mm}B
\end{bmatrix}.$$

Remark. The matrix $A \otimes B$ is called the Kronecker product of the matrices $A$ and $B$. ◀

2.28 Let $R$ be a domain with $Q = \text{Frac}(R)$, its field of fractions. If $A$ is an $R$-module, prove that every element in $Q \otimes_R A$ has the form $q \otimes a$ for $q \in Q$ and $a \in A$ (instead of $\sum_i q_i \otimes a_i$). (Compare this result with Example 2.67.)

*2.29 (i) Let $p$ be a prime, and let $p, q$ be relatively prime. Prove that if $A$ is a $p$-primary group and $a \in A$, then there exists $x \in A$ with $qx = a$.

(ii) If $D$ is a finite cyclic group of order $m$, prove that $D/nD$ is a cyclic group of order $d = (m, n)$.

(iii) Let $m$ and $n$ be positive integers, and let $d = (m, n)$. Prove that there is an isomorphism of abelian groups

$$\mathbb{I}_m \otimes \mathbb{I}_n \cong \mathbb{I}_d.$$  

(iv) Let $G$ and $H$ be finitely generated abelian groups, so that

$$G = A_1 \oplus \cdots \oplus A_n \quad \text{and} \quad H = B_1 \oplus \cdots \oplus B_m,$$

where $A_i$ and $B_j$ are cyclic groups. Compute $G \otimes_{\mathbb{Z}} H$ explicitly.

Hint. $G \otimes_{\mathbb{Z}} H \cong \sum_{i,j} A_i \otimes_{\mathbb{Z}} B_j$. If $A_i$ or $B_j$ is infinite cyclic, use Proposition 2.58; if both are finite, use part (ii).

*2.30 (i) Given $A_R$, $RB_S$, and $SC$, define $T(A, B, C) = F/N$, where $F$ is the free abelian group on all ordered triples $(a, b, c) \in A \times B \times C$, and $N$ is the subgroup generated by all

$$(ar, b, c) - (a, rb, c),$$
(a, bs, c) − (a, b, sc),

(a + a', b, c) − (a, b, c) − (a', b, c),

(a, b + b', c) − (a, b, c) − (a', b', c),

(a, b, c + c') − (a, b, c) − (a, b, c').

Define \( h: A \times B \times C \rightarrow T(A, B, C) \) by \( h: (a, b, c) \mapsto a \otimes b \otimes c \), where \( a \otimes b \otimes c = (a, b, c) + N \). Prove that this construction gives a solution to the universal mapping problem for triadditive functions.

**(ii)** Let \( R \) be a commutative ring and let \( A_1, \ldots, A_n, M \) be \( R \)-modules, where \( n \geq 2 \). An \( R \)-**multilinear function** is a function \( h: A_1 \times \cdots \times A_n \rightarrow M \) if \( h \) is additive in each variable (when we fix the other \( n - 1 \) variables), and \( f(a_1, \ldots, ra_i, \ldots, a_n) = rf(a_1, \ldots, a_i, \ldots, a_n) \) for all \( i \) and all \( r \in R \). Let \( F \) be the free \( R \)-module with basis \( A_1 \times \cdots \times A_n \), and define \( N \subseteq F \) to be the submodule generated by all the elements of the form

\[
(a_1, \ldots, ra_i, \ldots, a_n) - r(a_1, \ldots, a_i, \ldots, a_n)
\]

and

\[
(\ldots, a_i + a'_i, \ldots) - (\ldots, a_i, \ldots) - (\ldots, a'_i, \ldots).
\]

Define \( T(A_1, \ldots, A_n) = F/N \) and \( h: A_1 \times \cdots \times A_n \rightarrow T(A_1, \ldots, A_n) \) by \( (a_1, \ldots, a_n) \mapsto (a_1, \ldots, a_n) + N \). Prove that \( h \) is \( R \)-multilinear, and that \( h \) and \( T(A_1, \ldots, A_n) \) solve the universal mapping problem for \( R \)-multilinear functions.

**(iii)** Let \( R \) be a commutative ring and prove generalized associativity for tensor products of \( R \)-modules.

**Hint.** Prove that any association of \( A_1 \otimes \cdots \otimes A_n \) is also a solution to the universal mapping problem.

*2.31* Assume that the following diagram commutes, and that the vertical arrows are isomorphisms.

\[
\begin{array}{ccc}
0 & \longrightarrow & A' \\
\downarrow & & \downarrow \\
0 & \longrightarrow & A
\end{array}
\begin{array}{ccc}
& \longrightarrow & A \\
& & \downarrow \\
& \longrightarrow & A'' \\
& & \downarrow \\
& \longrightarrow & 0
\end{array}
\begin{array}{ccc}
& \longrightarrow & B' \\
& & \downarrow \\
& \longrightarrow & B \\
& & \downarrow \\
& \longrightarrow & B'' \\
& & \downarrow \\
& \longrightarrow & 0
\end{array}
\]

Prove that the bottom row is exact if and only if the top row is exact.
*2.32 (3 × 3 Lemma) Consider the following commutative diagram in $\mathit{RMod}$ having exact columns.

\[
\begin{array}{ccc}
0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & A' \to A \to A'' \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & B' \to B \to B'' \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & \to & C' \to C \to C'' \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

If the bottom two rows are exact, prove that the top row is exact; if the top two rows are exact, prove that the bottom row is exact.

*2.33 Consider the following commutative diagram in $\mathit{RMod}$ having exact rows and columns.

\[
\begin{array}{ccc}
A' & \to & A \to A'' \to 0 \\
\downarrow & \downarrow & \downarrow \\
B' & \to & B \to B'' \to 0 \\
\downarrow & \downarrow & \downarrow \\
C' & \to & C \to C'' \to 0 \\
\downarrow & \downarrow & \downarrow \\
0 & 0 & 0
\end{array}
\]

If $A'' \to B''$ and $B' \to B$ are injections, prove that $C' \to C$ is an injection. Similarly, if $C' \to C$ and $A \to B$ are injections, then $A'' \to B''$ is an injection. Conclude that if the last column and the second row are short exact sequences, then the third row is a short exact sequence and, similarly, if the bottom row and the second column are short exact sequences, then the third column is a short exact sequence.

2.34 Give an example of a commutative diagram with exact rows and vertical maps $h_1, h_2, h_4, h_5$ isomorphisms

\[
\begin{array}{cccc}
A_1 & \to & A_2 & \to & A_3 & \to & A_4 & \to & A_5 \\
\downarrow & & \downarrow & & \downarrow & & h_4 & & \downarrow & & h_5 \\
B_1 & \to & B_2 & \to & B_3 & \to & B_4 & \to & B_5
\end{array}
\]

for which there does not exist a map $h_3: A_3 \to B_3$ making the diagram commute.

*2.35 If $A, B,$ and $C$ are categories, then a bifunctor $T: A \times B \to C$ assigns, to each ordered pair of objects $(A, B)$, where $A \in \text{ob}(A)$ and $B \in \text{ob}(B)$, an object $T(A, B) \in \text{ob}(C)$, and to each ordered pair
of morphisms $f : A \to A'$ in $A$ and $g : B \to B'$ in $B$, a morphism $T(f, g) : T(A, B) \to T(A', B')$, such that

(a) fixing either variable is a functor; for example, if $A \in \text{ob}(A)$, then $T_A = T(A, \square) : B \to C$ is a functor, where $T_A(B) = T(A, B)$ and $T_A(g) = T(1_A, g)$.

(b) the following diagram commutes:

$$
\begin{array}{ccc}
T(A, B) & \xrightarrow{T(1_A, g)} & T(A, B') \\
\downarrow{T(f, 1_B)} & & \downarrow{T(f, 1_B')} \\
T(A', B) & \xrightarrow{T(1_{A'}, g)} & T(A', B')
\end{array}
$$

(i) Prove that $\otimes : \text{Mod}_R \times \text{Mod}_R \to \text{Ab}$ is a bifunctor.

(ii) Prove that $\text{Hom} : \text{Mod}_R \times \text{Mod}_R \to \text{Ab}$ is a bifunctor if we modify the definition of bifunctor to allow contravariance in one variable.

*2.36* Let $R$ be a commutative ring, and let $F$ be a free $R$-module.

(i) If $m$ is a maximal ideal in $R$, prove that $(R/m) \otimes_R F$ and $F/mF$ are isomorphic as vector spaces over $R/m$.

(ii) Prove that $\text{rank}(F) = \dim((R/m) \otimes_R F)$.

(iii) If $R$ is a domain with fraction field $Q$, prove that $\text{rank}(F) = \dim(Q \otimes_R F)$.

*2.37* Assume that a ring $R$ has IBN; that is, if $R^m \cong R^n$ as left $R$-modules, then $m = n$. Prove that if $R^m \cong R^n$ as right $R$-modules, then $m = n$.

**Hint.** If $R^m \cong R^n$ as right $R$-modules, apply $\text{Hom}_R(\square, R)$, using Proposition 2.54(iii).

*2.38* Let $R$ be a domain and let $A$ be an $R$-module.

(i) Prove that if the multiplication $\mu_r : A \to A$ is an injection for all $r \neq 0$, then $A$ is torsion-free; that is, there are no nonzero $a \in A$ and $r \in R$ with $ra = 0$.

(ii) Prove that if the multiplication $\mu_r : A \to A$ is a surjection for all $r \neq 0$, then $A$ is divisible.

(iii) Prove that if the multiplication $\mu_r : A \to A$ is an isomorphism for all $r \neq 0$, then $A$ is a vector space over $Q$, where $Q = \text{Frac}(R)$.

**Hint.** A module $A$ is a vector space over $Q$ if and only if it is torsion-free and divisible.

(iv) If either $C$ or $A$ is a vector space over $Q$, prove that both $C \otimes_R A$ and $\text{Hom}_R(C, A)$ are also vector spaces over $Q$. 