1. Let $R = k[X]/(X^3)$ where $k$ is a field. Let $C$ be the complex $R \xrightarrow{X^2} R$.
   (a) Find $\dim_k \text{Hom}_R^\text{Comp}(C, C)$, the dimension of the space of chain maps from $C$ to $C$.
   (b) Find the dimension of the subspace of chain maps $C \to C$ which are homotopic to zero. Hence find the dimension of the space $\text{Hom}_R^{\text{HoComp}}(C, C)$ of homotopy classes of chain maps $C \to C$.
   (c) Show that, for this complex $C$, the set of chain maps $C \to C$ which are non-isomorphisms forms a vector subspace of the space of all endomorphisms of $C$. Find the dimension of this subspace.
   (d) Show that it is possible to find another complex $D$ for which the set of non-isomorphisms $D \to D$ does not form a vector subspace of all endomorphisms.
   (e) Show that, for this complex $C$, all chain maps $C \to C$ which are equivalences are, in fact, automorphisms.
   (f) Determine, for this complex $C$, whether or not all invertible chain maps $C \to C$ are homotopic to each other.

2. (a) Suppose that $U$, $V$, and $W$ are $R$-modules and that there are homomorphisms

$$
\begin{array}{ccc}
U & \xrightarrow{\alpha} & V \\
\xleftarrow{\delta} & \beta & \xleftarrow{\gamma} W
\end{array}
$$

such that $\beta \alpha = 0$ and such that the identity map on $V$ can be written $1_V = \alpha \delta + \gamma \beta$. Show that $\beta = \beta \gamma \beta$. Suppose in addition to all this that $\alpha = \alpha \delta \alpha$. Show that $V \cong \alpha \delta (V) \oplus \gamma \beta (V)$.

(b) Recall that a chain complex $C$ of $R$-modules is called contractible if it is chain homotopy equivalent to the zero chain complex. Prove that $C$ is contractible if and only if $C$ can be written as a direct sum of chain complexes of the form $\cdots \to 0 \to A \xrightarrow{\alpha} B \to 0 \cdots$ where $\alpha$ is an isomorphism.
3. (a) Suppose that we have chain maps \( C \xrightarrow{f} D \xrightarrow{g} E \) and suppose that \( D \) is a contractible complex. Show that the composite \( gf \) is homotopic to zero (i.e. null homotopic).

(b) By considering the diagram

\[
\begin{array}{ccccccc}
C & \cdots & \overset{d}{\rightarrow} & C_2 & \overset{d}{\rightarrow} & C_1 & \overset{d}{\rightarrow} & C_0 & \overset{d}{\rightarrow} & \cdots \\
\downarrow{i_C} & & \downarrow{(d)} & & \downarrow{(d)} & & \downarrow{(d)} & & \\
I_C & : & \cdots & \overset{\delta}{\rightarrow} & C_1 \oplus C_2 & \overset{\delta}{\rightarrow} & C_0 \oplus C_1 & \overset{\delta}{\rightarrow} & C_1 \oplus C_0 & \overset{\delta}{\rightarrow} & \cdots \\
\end{array}
\]

where \( \delta = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \) show that any complex complex \( C \) can be embedded in a contractible complex \( I_C \).

(c) Show that if \( f = Td + eT : C \rightarrow D \) is any null-homotopic map of complexes then \( f \) defines a chain map \( I_C \rightarrow D \) as follows:

\[
\begin{array}{ccccccc}
C & \cdots & \overset{\delta}{\rightarrow} & C_1 \oplus C_2 & \overset{\delta}{\rightarrow} & C_0 \oplus C_1 & \overset{\delta}{\rightarrow} & C_1 \oplus C_0 & \overset{\delta}{\rightarrow} & \cdots \\
\downarrow & & \downarrow{(T,eT)} & & \downarrow{(T,eT)} & & \downarrow{(T,eT)} & & \\
D & : & \cdots & \overset{e}{\rightarrow} & D_2 & \overset{e}{\rightarrow} & D_1 & \overset{e}{\rightarrow} & D_0 & \overset{e}{\rightarrow} & \cdots \\
\end{array}
\]

such that the composite of this morphism with \( i_C \) is \( f \). Deduce that any null-homotopic map factors through a contractible complex.

**Extra questions: do not hand in!**

Given a homomorphism of chain complexes of \( R \)-modules \( \phi : C \rightarrow D \) we may define \( E_n = C_{n-1} \oplus D_n \), and a mapping \( e_n : E_n \rightarrow E_{n-1} \) by \( e_n(a, b) = (-\partial a, \phi a + \partial b) \), where we denote the boundary maps on \( C \) and \( D \) by \( \partial \). The specification \( \mathcal{E}(\phi) = \{ E_n, e_n \} \) is called the *mapping cone* of \( \phi \).

A. Show that \( \mathcal{E} = \{ E_n, e_n \} \) is indeed a chain complex.

B. Show that there is a short exact sequence of chain complexes \( 0 \rightarrow D \rightarrow \mathcal{E} \rightarrow C[1] \rightarrow 0 \) where \( C[1] \) denotes the chain complex with the same \( R \)-modules and boundary maps as \( C \) but with the labeling of degrees shifted by 1 in an appropriate direction. Deduce that there is a long exact sequence

\[
\cdots \rightarrow H_n(C) \rightarrow H_n(D) \rightarrow H_n(\mathcal{E}(\phi)) \rightarrow H_{n-1}(C) \rightarrow \cdots
\]

Show that \( \mathcal{E}(\phi) \) is acyclic if and only if \( \phi \) induces an isomorphism \( H_n(C) \rightarrow H_n(D) \) for every \( n \).

C. Show that if \( \phi \simeq \psi : C \rightarrow D \) then \( \mathcal{E}(\phi) \cong \mathcal{E}(\psi) \).
prove that the following diagram is commutative and has exact rows:

\[
\begin{array}{ccccccccc}
A'_n/\text{im} d'_{n+1} & \xrightarrow{i_n} & A_n/\text{im} d_{n+1} & \xrightarrow{p_n} & A''_n/\text{im} d''_{n+1} & \rightarrow & 0 \\
\downarrow d' & & \downarrow d & & \downarrow d'' & & \\
0 & \rightarrow & \ker d'_{n-1} & \rightarrow & \ker d_{n-1} & \rightarrow & \ker d''_{n-1}.
\end{array}
\]

(iv) Use part (ii) and this last diagram to give another proof of Theorem 6.10, the Long Exact Sequence.

6.6 Let \( f, g : C \rightarrow C' \) be chain maps, and let \( F : C \rightarrow C' \) be an additive functor. If \( f \simeq g \), prove that \( Ff \simeq Fg \); that is, if \( f \) and \( g \) are homotopic, then \( Ff \) and \( Fg \) are homotopic.

*6.7 Let \( 0 \rightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \rightarrow 0 \) be an exact sequence of complexes in which \( C' \) and \( C'' \) are acyclic; prove that \( C \) is also acyclic.

6.8 Let \( R \) and \( A \) be rings, and let \( T : R\text{Mod} \rightarrow A\text{Mod} \) be an exact additive functor. Prove that \( T \) commutes with homology; that is, for every complex \( (C, d) \in R\text{Comp} \) and for every \( n \in \mathbb{Z} \), there is an isomorphism

\[
H_n(TC,Td) \cong TH_n(C,d).
\]

*6.9 (i) Prove that homology commutes with direct sums: for all \( n \), there are natural isomorphisms

\[
H_n\left( \bigoplus_{\alpha} C^\alpha \right) \cong \bigoplus_{\alpha} H_n(C^\alpha).
\]

(ii) Define a direct system of complexes \( (C^i)_{i \in I}, (\varphi^i_j)_{i \leq j} \), and prove that \( \lim_{\rightarrow} C^i \) exists.

(iii) If \( (C^i)_{i \in I}, (\varphi^i_j)_{i \leq j} \) is a direct system of complexes over a directed index set, prove, for all \( n \geq 0 \), that

\[
H_n(\lim_{\rightarrow} C^i) \cong \lim_{\rightarrow} H_n(C^i).
\]

*6.10 Assume that a complex \( (C, d) \) of \( R \)-modules has a contracting homotopy in which the maps \( s_n : C_n \rightarrow C_{n+1} \) satisfying

\[
l_{C_n} = d_{n+1}s_n + s_{n-1}d_n
\]

are only \( \mathbb{Z} \)-maps. Prove that \( (C, d) \) is an exact sequence.

*6.11 \((\text{Barratt–Whitehead})\). Consider the commutative diagram with exact rows:

\[
\begin{array}{cccccccc}
\rightarrow & A_n & \xrightarrow{i_n} & B_n & \xrightarrow{p_n} & C_n & \xrightarrow{h_n} & A_{n-1} & \rightarrow & B_{n-1} & \rightarrow & C_{n-1} & \rightarrow & \\
\downarrow f_n & & \downarrow g_n & & \downarrow h_n & & \downarrow f_{n-1} & & \downarrow g_{n-1} & & \downarrow h_{n-1} & & \\
\rightarrow & A'_n & \xrightarrow{j_n} & B'_n & \xrightarrow{q_n} & C'_n & \xrightarrow{h_{n-1}} & A'_{n-1} & \rightarrow & B'_{n-1} & \rightarrow & C'_{n-1} & \rightarrow & .
\end{array}
\]
If each $h_n$ is an isomorphism, prove that there is an exact sequence

$$
\rightarrow A_n \xrightarrow{(f_n, i_n)} A'_n \oplus B_n \xrightarrow{j_n - g_n} B'_n \xrightarrow{\partial_n h_n^{-1} q_n} A_{n-1} \\
\rightarrow A'_{n-1} \oplus B_{n-1} \rightarrow B'_{n-1} \rightarrow ,
$$

where

$$(f_n, i_n): a_n \mapsto (f_n a_n, i_n a_n) \text{ and } j_n - g_n: (a'_n, b_n) \mapsto j_n a'_n - g_n b_n.$$

*6.12 (Mayer–Vietoris). Given a commutative diagram of complexes with exact rows,

$$
0 \rightarrow C' \xrightarrow{i} C \xrightarrow{p} C'' \rightarrow 0 \\
f \downarrow \quad g \downarrow \quad h \\
0 \rightarrow A' \xrightarrow{j} A \xrightarrow{q} A'' \rightarrow 0,
$$

if every third vertical map $h_a$ in the diagram

$$
\rightarrow H_n(C') \xrightarrow{i_a} H_n(C) \xrightarrow{p_a} H_n(C'') \xrightarrow{\partial_a} H_{n-1}(C') \rightarrow \\
f_a \downarrow \quad g_a \downarrow \quad h_a \downarrow \quad f_a \\
\rightarrow H_n(A') \xrightarrow{j_a} H_n(A) \xrightarrow{q_a} H_n(A'') \xrightarrow{\partial_a'} H_{n-1}(A') \rightarrow
$$
is an isomorphism, prove that there is an exact sequence

$$
\rightarrow H_n(C') \rightarrow H_n(A') \oplus H_n(C) \rightarrow H_n(A) \rightarrow H_{n-1}(C') \rightarrow .
$$

6.2 Derived Functors

In order to apply the general results in the previous section, we need a source of short exact sequences of complexes. The idea is to replace every module by a deleted resolution of it; given a short exact sequence of modules, we shall see that this replacement gives a short exact sequence of complexes. We then apply either Hom or $\otimes$, and the resulting homology modules are called Ext or Tor.

We know that a module has many presentations; since resolutions are generalized presentations, the next result is fundamental.

**Theorem 6.16 (Comparison Theorem).** Let $A$ be an abelian category. Given a morphism $f: A \rightarrow A'$ in $A$, consider the diagram

$$
\begin{array}{ccccccccc}
P_2 & \xrightarrow{d_2} & P_1 & \xrightarrow{d_1} & P_0 & \xrightarrow{\epsilon} & A & \rightarrow & 0 \\
| & \downarrow{f_2} & | & \downarrow{f_1} & | & \downarrow{f_0} & | & \downarrow{f} \\
P'_2 & \xrightarrow{d'_2} & P'_1 & \xrightarrow{d'_1} & P'_0 & \xrightarrow{\epsilon'} & A' & \rightarrow & 0,
\end{array}
$$
where $\partial_n = \sum_{i=0}^n (-1)^i d_i^n$. The given identities for $d_i^n$ imply $\partial \partial = 0$. Thus, simplicial objects have homology. The degeneracies allow one to construct an abstract version of homotopy groups as well (see Gelfand–Manin, *Methods of Homological Algebra*, May, *Simplicial Objects in Algebraic Topology*, and Weibel, *An Introduction to Homological Algebra*).

**Exercises**

6.13 If $\tau: F \to G$ is a natural transformation between additive functors, prove that $\tau$ gives chain maps $\tau_C: FC \to GC$ for every complex $C$. If $\tau$ is a natural isomorphism, prove that $FC \cong GC$.

*6.14* Consider the commutative diagram with exact row

$$
\begin{array}{ccc}
B' & \xrightarrow{f} & C & \xrightarrow{q} & B'' \\
\downarrow{k} & & \downarrow{\ell} & & \downarrow{p} \\
B & & & & \\
\end{array}
$$

If $k$ is an isomorphism with inverse $\ell$, prove exactness of

$$
B' \xrightarrow{i} B \xrightarrow{p} B''
$$

6.15 Let $T: A \to C$ be an exact additive functor between abelian categories, and suppose that $P$ projective implies $TP$ projective. If $B \in \text{obj}(A)$ and $P_B$ is a deleted projective resolution of $B$, prove that $TP_B$ is a deleted projective resolution of $TB$.

6.16 Let $R$ be a $k$-algebra, where $k$ is a commutative ring, which is flat as a $k$-module. Prove that if $B$ is an $R$-module (and hence a $k$-module), then

$$
R \otimes_k \text{Tor}_n^R(B, C) \cong \text{Tor}_n^R(B, R \otimes_k C)
$$

for all $k$-modules $C$ and all $n \geq 0$.

6.17 Let $R$ be a semisimple ring.

(i) Prove, for all $n \geq 1$, that $\text{Tor}_n^R(A, B) = \{0\}$ for all right $R$-modules $A$ and all left $R$-modules $B$.

(ii) Prove, for all $n \geq 1$, that $\text{Ext}_n^R(A, B) = \{0\}$ for all left $R$-modules $A$ and $B$.

*6.18* If $R$ is a PID, prove, for all $n \geq 2$, that $\text{Tor}_n^R(A, B) = \{0\} = \text{Ext}_n^R(A, B)$ for all $R$-modules $A$ and $B$.

**Hint.** Use Corollary 4.15.

*6.19* Let $R$ be a domain with fraction field $Q$, and let $A$, $C$ be $R$-modules. If either $C$ or $A$ is a vector space over $Q$, prove that $\text{Tor}_n^R(C, A)$ and $\text{Ext}_n^R(C, A)$ are also vector spaces over $Q$.

**Hint.** Use Exercise 2.38 on page 97.