Math 8212 **Homework 1** **PJW**

Date due: Wednesday February 6, 2013. In class on Friday February 8 we will grade your answers, so it is important to be present on that day, with your homework.

Questions 1.1 - 1.6 on page 6 of Matsumura.

Extra questions (not part of the assignment):

As in the book, A is a commutative ring with a 1.

1. Write out a proof that when k is a field, the Jacobson radical of $k[x_1, \ldots, x_n]$ is 0.
2. (From page 3) Write out a proof that that if $1 + Ax$ consists of units then $x \in \operatorname{Rad} A$.
3. Assume that A is an integral domain and let $a \in A$. Show that a is irreducible if and only if (a) is maximal among proper principal ideals
4. Find a ring A and a maximal ideal I of $A[X]$ so that $I \cap A$ is not a maximal ideal of A.

$A = \mathbb{Z}[\sqrt{-5}] = \mathbb{Z}[X]/(X^2+5)$; then setting $k = \mathbb{Z}/2\mathbb{Z}$ we have

$$A/2A = \mathbb{Z}[X]/(2, X^2+5) = k[X]/(X^2-1) = k[X]/(X-1)^2.$$

Then $P = (2, 1-\sqrt{-5})$ is a maximal ideal of A containing 2.

Exercises to §1. Prove the following propositions.

1.1. Let A be a ring, and $I \subset \text{nil}(A)$ an ideal made up of nilpotent elements; if $a \in A$ maps to a unit of A/I then a is a unit of A.

1.2. Let $A_1, \ldots, A_n$ be rings; then the prime ideals of $A_1 \times \cdots \times A_n$ are of the form
$A_1 \times \cdots \times A_{i-1} \times P_i \times A_{i+1} \times \cdots \times A_n$,
where P_i is a prime ideal of A_i.

1.3. Let A and B be rings, and $f: A \longrightarrow B$ a surjective homomorphism.
(*a*) Prove that $f(\text{rad}\, A) \subset \text{rad}\, B$, and construct an example where the inclusion is strict.
(*b*) Prove that if A is a semilocal ring then $f(\text{rad}\, A) = \text{rad}\, B$.

1.4. Let A be an integral domain. Then A is a UFD if and only if every irreducible element is prime and the principal ideals of A satisfy the ascending chain condition. (Equivalently, every non-empty family of principal ideals has a maximal element.)

1.5. Let $\{P_\lambda\}_{\lambda \in \Lambda}$ be a non-empty family of prime ideals, and suppose that the P_λ are totally ordered by inclusion; then $\bigcap P_\lambda$ is a prime ideal. Also, if I is any proper ideal, the set of prime ideals containing I has a minimal element.

1.6. Let A be a ring, $I, P_1, \ldots, P_r$ ideals of A, and suppose that $P_3, \ldots, P_r$ are prime, and that I is not contained in any of the P_i; then there exists an element $x \in I$ not contained in any P_i.

2 Modules

Let A be a ring and M an A-module. Given submodules N, N' of M, the set $\{a \in A \mid aN' \subset N\}$ is an ideal of A, which we write $N:N'$ or $(N:N')_A$. Similarly, if $I \subset A$ is an ideal then $\{x \in M \mid Ix \subset N\}$ is a submodule of M, which we write $N:I$ or $(N:I)_M$. For $a \in A$ we define $N:a$ similarly. The ideal $0:M$ is called the *annihilator* of M, and written $\text{ann}(M)$. We can consider M as a module over $A/\text{ann}(M)$. If $\text{ann}(M) = 0$ we say that M is a *faithful* A-module. For $x \in M$ we write $\text{ann}(x) = \{a \in A \mid ax = 0\}$.

If M and M' are A-modules, the set of A-linear maps from M to M' is written $\text{Hom}_A(M, M')$. This becomes an A-module if we define the sum $f+g$ and the scalar product af by

$$(f+g)(x) = f(x) + g(x), \quad (af)(x) = a \cdot f(x);$$

(the fact that af is A-linear depends on A being commutative).