Equations of lines and planes

We are familiar with various forms for the equation of a line in $\mathbb{R}^2$. There is the slope–intercept form

$$y = mx + c$$

where m is the slope and c is the intercept with the y-axis. This works as long as the line is not parallel to the y-axis, in which case we should use some other form of the equation such as

$$ax + by = c$$

or possibly

$$\frac{x-u}{s} = \frac{y-v}{t}$$

which gives a line passing through (u, v). We could also describe the line parametrically, as the set of points $\underline{u}+t\underline{v}$ where t is allowed to vary through $\mathbb{R}$, for a line passing through $\underline{u}$, in the direction of $\underline{v}$. To be able to convert between this forms of the equations is important.

In $\mathbb{R}^3$ a line needs two equations to define it, such as

$$\frac{x_1-u_1}{w_1} = \frac{x_2-u_2}{w_2} = \frac{x_3-u_3}{w_3}$$

for a line passing through the point (u_1, u_2, u_3). We can also describe a line parametrically as $\underline{u}+t\underline{v}$ where t is allowed to vary through $\mathbb{R}$, for a line passing through $\underline{u}$, in the direction of $\underline{v}$ (the same as with $\mathbb{R}^2$, except now the vectors are in $\mathbb{R}^3$). It is important to be able to convert between these descriptions of a line, and we should be able to find some point lying on a line, and also a vector pointing in the direction of the line, no matter how the line is given.

Exercises: 1. Express the line given parametrically as $(1, 1, 1) + t(2, 3, 5)$ in the form above where there are two equations.

2. Find a parametric form for the line

$$x - 1 = y + 2 = \frac{z-3}{2}.$$

3. To do 2. you will have solved the problems: find a point on the line given in 2; find a vector pointing in the direction of the line given in 2. These are legitimate questions in their own right.

A plane in $\mathbb{R}^3$ is specified by a single equation $ax + by + cz = d$, such as the plane

$$P_1 : \quad 5x - y + 2z = 4$$

We realize that this plane will pass through the origin if and only if $d = 0$, and in general the plane $ax + by + cz = d$ is parallel to the plane $ax + by + cz = 0$. Furthermore, the vector (a, b, c) is perpendicular, or *normal*, to the plane. This is because the equation of the plane $ax + by + cz = 0$ can be written $(a, b, c) \cdot (x, y, w) = 0$, so that the condition on (x, y, z) is that it is a vector perpendicular to (a, b, c).

More equations of lines and planes in $\mathbb{R}^3$

We are now in a position to solve various problems to do with points, lines and planes. Here is a comprehensive set of examples.

1. Find a vector perpendicular to the plane P_1 specified previously. Solution: $(5,-1,2)$.

2. Find a vector parallel to P_1. Solution: take two points in P_1 and compute their difference. There are many possible solutions. For example, taking $x = z = 0$ gives $y = -4$, so $(0,-4,0)$ lies in P_1. Similarly taking $x = y = 0$ gives $z = 2$, so $(0,0,2)$ lies in the plane. Now $(0,0,2) - (0,-4,0) = (0,4,2)$ is parallel to P_1.

3. Find a vector which points in the direction of the line of intersection of P_1 and the plane

$$P_2: \quad x + y - z = 1.$$

Solution: such a vector is perpendicular to the two normal vectors $(5,-1,2)$ and $(1,1,-1)$. Take their cross product.

4. Find a point lying on both P_1 and P_2. Solution: Solve the equations of the line simultaneously. There will be infinitely many solutions. We have $5x - y + 2z = 4$ and $x + y - z = 1$. Adding the two equations gives $6x + z = 5$. We could take $x = 0$ and $z = 5$. From the first equation we get $y = 5x + 2z - 4 = 6$. Thus $(0,6,5)$ lies on both P_1 and P_2.

5. Find the equations of the line of intersection of P_1 and P_2. Solution. Recall that two equations define a line, hence the plural. We have found a point on this line and a vector pointing in its direction, so we can apply what we already have done in the previous section of these notes.

6. Find the angle between the planes P_1 and P_2. Solution: We must first think what we mean by this angle. After deciding on that you will probably realize that it is the same as the angle between their normal vectors, which we have already seen can be taken to be $(5,-1,2)$ and $(1,1,-1)$. The cosine of the angle between them is the dot product of these vectors divided by the product of their lengths.

7. Find the equation of the plane passing through $(2,3,5)$ which is perpendicular to $(-1,-4,1)$. Solution: it has the form $-x - 4y + z = d$ for some number d. Substituting $2,3,5$ for x,y,z gives $-2 - 12 + 5 = -9 = d$, so the equations is $-x - 4y + z = -9$.

8. Find the equation of the plane parallel to P_1 which passes through $(1,-1,1)$. Solution: you can probably see how to do this by now. The plane is $5x - y + 2z = 8$.

9. Find the equation of the plane which passes through the points $(1,1,0)$, $(0,2,3)$ and $(0,0,2)$.

Solution: We first find a normal vector to the plane. We do this by finding two vectors parallel to the plane, such as $(1,1,0) - (0,2,3) = (1,-1,-3)$ and $(1,1,0) - (0,0,2) = (1,1,-2)$ and take their cross product. This gives the left side of the desired equation. We find the right side by substituting values of x,y,z from a point on the plane, as before.

Computing distances between points, lines and planes

You will be given questions which ask things like: find the shortest distance between the following two lines. This means the shortest possible distance between a pair of points, one lying on one line, the second lying on the other line.

10. Find the shortest distance from the point $(1,1,1)$ to the plane P_1. Solution: We find a point on P_1, such as $(0,0,2)$ and compute the vector between this point and $(1,1,1)$, namely $(1,1,1)-(0,0,2)=(1,1,-1)$. We now project this vector into the normal direction to P_1, namely $(5,-1,2)$. This projection is

$$\frac{(1,1,-1)\cdot(5,-1,2)}{|(5,-1,2)|}=\frac{5-1-2}{\sqrt{5^2+(-1)^2+2^2}}=\frac{2}{\sqrt{30}}.$$

11. Find the shortest distance from the point $(1,1,1)$ to the line $x-1=y+2=\frac{z-3}{2}$. Solution: Find a vector $\underline{v}$ in the direction of the line, such as $\underline{v}=(2,2,1)$. Find a point on the line, such as $(3,0,7)$. Find the vector $\underline{w}$ from $(3,0,7)$ to $(1,1,1)$, so $\underline{w}=(-2,1,-6)$. If θ is the angle between $\underline{v}$ and $\underline{w}$ then the shortest distance is $|\underline{w}|\sin\theta$. This equals

$$\frac{|\underline{v}\times\underline{w}|}{|\underline{v}|}$$

12. Find the shortest distance between the two lines $x-1=y+2=\frac{z-3}{2}$ and $x+2=2y+3=z$. Solution: This is perhaps the most complicated of these problems. Find vectors $\underline{u}$ and $\underline{v}$ pointing in the direction of the lines. Find a point on each line, and hence the vector $\underline{w}$ joining them. Now the shortest distance is the projection of this vector $\underline{w}$ into the direction perpendicular to both $\underline{u}$ and $\underline{v}$, because the line segment of shortest distance which joins the two lines is perpendicular to both lines. The perpendicular direction is obtained as $\underline{u}\times\underline{v}$. This projection is

$$\frac{\underline{w}\cdot(\underline{u}\times\underline{v})}{|\underline{u}\times\underline{v}|}.$$