Math 3592 review for exam 2

This sheet is supposed to help you by providing some extra practice. It should not be interpreted that what is on this sheet is the only practice you should do, or that questions like the ones here are the only kind you will be asked. You are advised to review the material we have covered more broadly than what is on this sheet.

At the top of Exam 2 it says:

There are 5 questions, each worth 20% of the total. You may not use books or notes. You may use a calculator. Always show your work. If you are not sure what is required in any question, or what the question means, do ask.

1. Let S be a subset of $\mathbb{R}$ and $x = \operatorname{Inf} S$. Write out a proof of the following assertion: for every $\epsilon > 0$ there exists a point $y \in S$ with $|x - y| < \epsilon$.

2. For each of the following functions $\mathbb{R}^2 \to \mathbb{R}$ find (a) the points at which it is continuous, (b) the points at which it is differentiable, and (c) the points at which all partial derivatives exist. See also questions 1.27 and 1.33 from section 1.10.

   (i)
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{xy}{x^2-y^2} & \text{if } x \neq \pm y, \\ 0 & \text{if } x = \pm y. \end{cases}$$

   (ii)
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{x|y|}{\sqrt{x^2+y^2}} & \text{if } \binom{x}{y} \neq \binom{0}{0}, \\ 0 & \text{if } \binom{x}{y} = \binom{0}{0}. \end{cases}$$

   (iii)
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{xy}{\sqrt{x^2+y^2}} & \text{if } \binom{x}{y} \neq \binom{0}{0}, \\ 0 & \text{if } \binom{x}{y} = \binom{0}{0}. \end{cases}$$

   (iv)
$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{cases} \frac{x^2}{\sqrt{x^2+y^2}} & \text{if } \binom{x}{y} \neq \binom{0}{0}, \\ 0 & \text{if } \binom{x}{y} = \binom{0}{0}. \end{cases}$$

3. If any of the above functions $f\binom{x}{y}$ are continuous at $\binom{0}{0}$, find a positive number $\delta > 0$ so that
$$\left|\begin{pmatrix} x \\ y \end{pmatrix}\right| < \delta \quad \text{implies} \quad \left|f\begin{pmatrix} x \\ y \end{pmatrix}\right| < \frac{1}{100}.$$

4. (See also Assignment 8 questions 2,3,4) Suppose that
$$z = xy^2, \quad \frac{dx}{dt} = \frac{1}{\sqrt{4+t^3}}, \quad \frac{dy}{dt} = e^t\sqrt{4+t}, \quad x(0) = 5 \text{ and } y(0) = 2.$$

   Find $\frac{dz}{dt}$ at $t = 0$.

5. Let A be a 2×2 real matrix and let I be the 2×2 identity matrix. Calculate the effect of the derivative of $f(A) = (I + A^2)^{-1}$ at $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ on the matrix $\begin{pmatrix} 1 & -1 \\ 2 & 0 \end{pmatrix}$.

6. Let $f(z) = 10z^{10} + 9z^9 + 8z^8 + \cdots + 2z^2 + z$.
   (i) Find $\delta > 0$ so that $|z| < \delta$ implies $|f(z)| < \frac{1}{10}$.
   (ii) Find N so that $|f(z)| > 100$ for all z with $|z| > N$.

7. Suppose that $f : \mathbb{R}^3 \to \mathbb{R}^4$ is a differentiable function and that $a \in \mathbb{R}^3$ is a point for which $Df(a) \neq 0$. Write out a proof that there exists a point $b \in \mathbb{R}^3$ such that $f(a) \neq f(b)$.

See also questions 1.8.9, 1.8.10, 1.8.11, and from section 1.10 questions 1.23, 1.24, 1.30, 1.31.