Math 8245 Homework 5 PJW

Date due: December 14, 2015. We will go over these questions in class on Dec. 16.

1. By drawing 3 planes cutting a regular cube so that the reflections in these planes generate the group of all isometries of the cube, identify this group of isometries as a Coxeter group. Draw the Coxeter diagram of the group. How big is the group?

2. By considering the effect of the element \( s_1 s_2 s_1 s_3 \) on the geometric representation show that in the group \( \langle s_1, s_2, s_3 \mid s_1^2, s_1^2, s_3^2, (s_1 s_2)^3, (s_1 s_3)^3, (s_2 s_3)^3 \rangle \) this element has infinite order.

3. Let \( e_i \) be the \( i \)th unit coordinate vector in \( \mathbb{R}^n \).
   (a) Show that the root system for the Coxeter group \( W \) whose diagram has \( n-1 \) nodes \( \circ \cdots \circ \) in a line may be identified with the \( n(n-1) \) vectors \( e_i - e_j \) with \( i \neq j \) in such a way that the simple roots (the vectors \( \alpha_s \)) are the \( e_2 - e_1, e_3 - e_2, \ldots, e_n - e_{n-1} \) in the hyperplane of vectors with coordinate sum zero.
   (b) Identify which of the vectors \( e_i - e_j \) are positive roots and which are negative roots.
   (c) By considering the action of \( W \) on the standard \( n-1 \)-simplex which is the convex hull of \( e_1, \ldots, e_n \) in \( \mathbb{R}^n \), show that \( W \cong S_n \).
   (d) Letting \( S_n \) act on \( \{1, \ldots, n\} \) in the usual way, show that if \( g \in S_n \) then \( \ell(g) \) equals the number of pairs \( i < j \) for which \( gi > gj \).

4. (Exercise 1 on p. 115 of Humphreys) Given a reduced expression \( w = s_1 \cdots s_r \) (\( s_i \in S \)), set \( \alpha_i := \alpha_{s_i} \) and \( \beta_i := s_{s_i} s_{s_i - 1} \cdots s_{i+1} (\alpha_i) \), interpreting \( \beta_r \) to be \( \alpha_r \). Prove that \( \Pi(w) \) (i.e. the set of positive roots sent to negative roots by \( w \)) consists of the \( r \) distinct positive roots \( \beta_1, \ldots, \beta_r \).

5. (Exercise 2 on p. 115 of Humphreys) (a) If \( W \) is infinite, prove that the length function takes arbitrarily large values, hence that \( \Phi \) is infinite. Show that the scalar \( -1 \in GL(V) \) does not lie in \( \sigma(W) \).
   (b) If \( W \) is finite, prove that there is one and only one element \( w_\circ \in W \) of maximal length, and that \( w_\circ \) maps \( \Pi \) onto \( -\Pi \).
   (c) Let \( S_n \) act on \( \{1, \ldots, n\} \) in the usual way. Show that \( w_\circ = (1, n)(2, n-1)(3, n-2) \cdots \).

6. (Exercise on p. 127 of Humphreys) If the Tits cone \( U \) is equal to \( V^* \), prove that \( W \) is finite. [Find \( w \in W \) for which \( w(\bar{C}) \) meets \( -C \). Then show that \( w^{-1}(\alpha_s) < 0 \) for all \( s \in S \), and deduce that \( W \) is finite.]
7. The Coxeter complex of a Coxeter system \((W, S)\) is a ‘simplicial complex’ whose simplices are the regions \(wC_I\) where \(w \in W\) and \(I\) ranges over the subsets of \(S\) which are not equal to the whole of \(S\). Here

\[
C_I = \{ f \in V^* \mid \langle f, \alpha_{s_i} \rangle = 0 \text{ for all } i \in I, \langle f, \alpha_{s_i} \rangle \geq 0 \text{ for all } i \not\in I \}.
\]

The subsimplices of \(wC_I\) are the \(w'C_I\) where \(w'C_I \subseteq wC_I\).

This construction really constructs the Coxeter complex in a way similar to an abstract simplicial complex. The set of simplices in each dimension is an abstract set, and face relationships are understood between simplices of different dimensions. Here the set of vertices, for instance, is a set of lines in \(V^*\).

(a) Show that \(w'C_J \subseteq wC_I\) if and only if \(J \supset I\) and \(w'C_J = wC_J\). [Assume the theorem in section 5.13 of Humphreys. In particular, note that \(C_I\) is the fixed point set \(D^W_I\).]

(b) Show that the Coxeter complex may also be constructed in the following way: we take the simplices to be in bijection with all cosets \(wW_I\) where \(w \in W\) and \(I \subset S\) is not equal to the whole of \(S\). The subsimplices of \(wW_I\) are the cosets \(w'W_J\) for which \(w'W_J \supseteq wW_I\).

(c) With this description of the Coxeter complex, show that two simplices \(wW_I\) and \(w'W_J\) meet in a (non-empty) Simlex if and only if there is a third coset \(w''W_K\) which contains them both. Show that in this case \(I \cup J \subseteq K\) and \(w''W_K = wW_K = w'W_K\). Identify which cosets are the vertices of the Coxeter complex, and also which cosets are the simplices of maximal dimension.

(d) Assume without proof that the maximal dimension of a simplex in the Coxeter complex is \(|S| - 1\), and that every simplex is contained in a simplex of this dimension. How many simplices of dimension \(|S| - 1\) contain each simplex of dimension \(|S| - 2\)?