Math 8246 **Homework 4** **PJW**

Date due: Monday May 2, 2016. We will discuss these questions on Wednesday 5/4/2016

1. (a) Calculate the number of ordered pairs (g_1, g_2) of elements of S_3 such that $\langle g_1, g_2\rangle = S_3$.

   (b) Find a minimal set of pairs (g_1, g_2) of elements of S_3 such that they generate S_3 and any generating pair (h_1, h_2) can be written $(h_1, h_2) = (\alpha g_1, \alpha g_2)$ for some $\alpha \in \mathrm{Aut}(S_3)$ and some pair (g_1, g_2) in the minimal set.

   (c) Show that $S_3 \times S_3$ can be generated by two elements, but that $S_3 \times S_3 \times S_3$ cannot.

2. Let Ω be a transitive G-set and let k be a field of positive characteristic p. Show that if p does not divide $|\Omega|$ then the permutation module $k\Omega$ decomposes as $k\Omega = k \oplus U$ as kG-modules, where U is a module with zero fixed points. Show by example that if $p \mid |\Omega|$ then $k\Omega$ may be indecomposable.

3. Let $G = S_3$ and let $k = \mathbb{F}_4$.

   (a) Show that $k \uparrow_{C_2}^{G} = k \oplus P_2$ as kG-modules, where P_2 is a 2-dimensional simple (or irreducible) module.

   (b) Show that $k \uparrow_{C_3}^{G} = P_1$ is an indecomposable module that is not simple, and that its composition factors are two copies of k.

   (c) Show that $P_1 \downarrow_{C_2}^{G} \cong P_2 \downarrow_{C_2}^{G}$ as kG-modules.

   (d) Decompose $k \uparrow_{C_2}^{G} \otimes_k k \uparrow_{C_2}^{G}$ as a direct sum of indecomposable modules (up to isomorphism).

   (e) Decompose $P_1 \otimes_k P_1$, $P_1 \otimes_k P_2$, and $P_2 \otimes_k P_2$ as direct sums of indecomposable modules.

4. Show that if H is a group that is cyclic mod p, then every subgroup of H is cyclic mod p.

5. (a) Make a list of the subgroups of A_5 that are cyclic mod 2.

   (b) Obtain an expression for the trivial module $\mathbb{F}_2$ as a linear combination of permutation modules with stabilizers that are cyclic mod 2.

   (c) For every (finitely gnerated) $\mathbb{Z}G$-module M, show that

   $$H^n(A_5, M)_2 \cong H^n(A_4, M)_2$$

   for all $n \geq 1$.

   (d) Give a formula for $H^0(A_5, M)$ in terms of the fixed points of A_4, C_5, and C_3.

6. Let H and K be subgroups of G. Show that $|N_G(H, K)|/|N_G(H)|$ equals the number of conjugates of H contained in K. Show also that $|N_G(H, K)|/|N_G(K)|$ equals the number of conjugates of K containing H.

7. Compute the table of marks of A_5.