Math 8300 **Examples 2** **PJW**

Date due: Monday February 27, 2017. We will discuss these questions on Wednesday 3/1/2017

These questions can all be done using technology presented in class. It would be possible to do some of them in a different way, perhaps by studying various texts. The point about these questions is that they reinforce what is done in class, and I prefer it if you use the methods I have taught.

1. (4 pts) Show by example that the homomorphism $FGL(E) \to S_F(n,r)$ given by the representation of $GL(E)$ on $E^{\otimes r}$ need not be surjective if the field F is not infinite.

2. (4 pts) Show by example that it is possible to find a group G, a $\mathbb{Z}G$-module U and a prime p so that the ring homomorphism $\mathrm{End}_{\mathbb{Z}G} \to \mathrm{End}_{\mathbb{F}_pG}(U/pU)$ is not surjective.

3. (2 pts) Let M be a module for a ring A, and suppose that M has just two composition factors and is indecomposable. Show that M has a unique submodule, other than 0 and M.

4. True or false? Provide either a proof or a counterexample for each part. Let t be a λ-tableau.
   (a) (2 pts) In any direct sum decomposition of M^λ as a direct sum of indecomposable $\mathbb{F}_pS_r$-modules, there is a unique summand on which κ_t has non-zero action.
   (b) (2 pts) Furthermore, if Y^μ is a Young module for $\mathbb{F}_pS_r$ which has a submodule isomorphic to S^λ then $\lambda \trianglerighteq \mu$.
   (c) (2 pts) Determine whether or not this gives a proof that the various Young modules Y^λ, as λ ranges through partitions of r, are all non-isomorphic.

5. In this question, tableaux may have repeated entries. Let λ be a partition of r, and let μ be any sequence of non-negative integers, whose sum is r. We say that a λ-tableau T has type μ if, for every i, the number i occurs μ_i times in T. For example, $\begin{matrix} 2 & 2 & 1 & 1 \\ 1 \end{matrix}$ is a $[4,1]$-tableau of type $[3,2]$. We will number the positions in T according to some tableau with distinct entries, such as

$$t = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 \end{matrix},$$

but it could have been some other such tableau.
   (a) (2 pts) Show that the set of λ-tableaux of type μ is in bijection with the set of μ-tabloids.

We now let S_r act on the λ-tableaux of type μ by permuting the positions of the entries. Thus if $T = \begin{matrix} 2 & 2 & 1 & 1 \\ 1 \end{matrix}$ then $T(1,5) = \begin{matrix} 1 & 2 & 1 & 1 \\ 2 \end{matrix}$ and $T(1,5,2) = \begin{matrix} 2 & 1 & 1 & 1 \\ 2 \end{matrix}$ since $(1,5,2) = (1,5)(1,2)$. We say that T_1 and T_2 are row equivalent if $T_2 = T_1\pi$ for some permutation in the row stabilizer of the λ-tableau t.

(b) (2 pts) Show that the row equivalence classes of λ-tableaux of type μ are in bijection with the double cosets $S_\mu \backslash S_r / S_\lambda$.

(c) (2 pts) Show that for each λ-tableau T of type μ there is a RS_r-module homomorphism $\theta_T : M^\lambda \to M^\mu$ such that $\theta_T(\{t\}) = \sum\{T_i \mid T_i \text{ is row equivalent to } T\}$. Thus, in the above example,

$$\theta_T(\{t\}) = \begin{matrix} 2 & 2 & 1 & 1 \\ 1 \end{matrix} + \begin{matrix} 2 & 1 & 2 & 1 \\ 1 \end{matrix} + \begin{matrix} 2 & 1 & 1 & 2 \\ 1 \end{matrix}$$
$$+ \begin{matrix} 1 & 2 & 2 & 1 \\ 1 \end{matrix} + \begin{matrix} 1 & 2 & 1 & 2 \\ 1 \end{matrix} + \begin{matrix} 1 & 1 & 2 & 2 \\ 1 \end{matrix}.$$

(d) (2 pts) Show that, as T ranges over the row equivalence classes of λ-tableaux of type μ the homomorphisms θ_T give a basis for $\mathrm{Hom}_{RS_r}(M^\lambda, M^\mu)$.

6. In this question you may assume that there is a decomposition of the group algebra $\mathbb{F}_2 S_3 \cong Y^{[1^3]} \oplus Y^{[2,1]} \oplus Y^{[2,1]}$ and that $Y^{[1^3]}$ has dimension 2, and has a unique $\mathbb{F}_2 S_3$-submodule of dimension 1. Let $E = \mathbb{F}_2^3$ be a 3-dimensional space over $\mathbb{F}_2$.
   (a) (2 pts) Express $E^{\otimes 3}$ as a direct sum of modules M^λ, determining the multiplicity of each M^λ summand.
   (b) (2 pts) Make a table with rows and columns indexed by the partitions of 3, whose λ, μ entry is the number of double cosets $|S_\lambda \backslash S_3 / S_\mu|$.
   (c) (2 pts) Compute the dimension of $S_{\mathbb{F}_2}(3,3)$.
   (d) (2 pts) Compute the dimensions of the simple modules for $S_{\mathbb{F}_2}(3,3)$.
   (d) (2 pts) Compute a list of the composition factors of each indecomposable projective $S_{\mathbb{F}_2}(3,3)$-module, assuming the projective has the form $S_{\mathbb{F}_2}(3,3)e$ for some idempotent e.
   (e) (2 pts) Show that, as $S_{\mathbb{F}_2}(3,3)$-modules, the symmetric tensors $ST^3(E)$ is indecomposable, but that $E^{\otimes 3}$ is the direct sum of three indecomposable submodules, and find their dimensions.

Extra question: do not hand in

7. Find a basis for the space of homomorphisms $\mathrm{Hom}_{FS_5}(M^{(3,2)}, M^{(2,1,1,1)})$. For each element θ in your basis, compute the effect of θ on the tabloid

$$\begin{array}{c} \hline 1 \; 2 \; 3 \\ \hline 4 \; 5 \\ \hline \end{array}.$$

8. Let $U = U_1 \oplus U_2 = U_1' \oplus U_2'$ be two direct sum decompositions of a module for an algebra A, and let $1_U = f_1 + f_2 = f_1' + f_2'$ be the corresponding expressions for 1_u as sums of orthogonal idempotents in $\mathrm{End}_A(U)$. Show that (a) $U_1 \cong U_1'$ and $U_2 \cong U_2'$ as A-modules, if and only if (b) there exists an invertible $\alpha \in \mathrm{End}_A(U)$ so that $f_1' = \alpha f_1 \alpha^{-1}$, if and only if (c) as $\mathrm{End}_A(U)$-modules, $\mathrm{End}_A(U)f_1 \cong \mathrm{End}_A(U)f_1'$ and $\mathrm{End}_A(U)f_2 \cong \mathrm{End}_A(U)f_2'$.