1. (4 pts) Show by example that the homomorphism $FGL(E) \to S_F(n,r)$ given by the representation of $GL(E)$ on $E^{\otimes r}$ need not be surjective if the field F is not infinite.

Solution: We have computed in class that $\dim S_{\mathbb{F}_2}(2,2) = 10$. On the other hand, $|GL(2,2)| = 6$ and this is the dimension of $\mathbb{F}_2GL(E)$. Since $6 < 10$ the map is not surjective.

2. (4 pts) Show by example that it is possible to find a group G, a $\mathbb{Z}G$-module U and a prime p so that the ring homomorphism $\mathrm{End}_{\mathbb{Z}G} \to \mathrm{End}_{\mathbb{F}_pG}(U/pU)$ is not surjective.

Solution: Let $G = C_2 = \langle g\rangle$ and $U = \mathbb{Z}\oplus\mathbb{Z}$ with g acting via $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. Then $\mathrm{End}_{\mathbb{Z}G}(U) \cong \mathbb{Z}^2$, but $\mathrm{End}_{\mathbb{F}_2G}(U/2U) \cong \mathbb{F}_2^4$.

3. (2 pts) Let M be a module for a ring A, and suppose that M has just two composition factors and is indecomposable. Show that M has a unique submodule, other than 0 and M.

Solution: If M has two distinct proper submodules U, V they must both be simple and so $U \cap V = 0$ because this is a proper submodule of a simple module. Now $U + V = U \oplus V$ has composition length 2, so $U \oplus V = M$.

4. True or false? Provide either a proof or a counterexample for each part. Let t be a λ-tableau.
   (a) (2 pts) In any direct sum decomposition of M^λ as a direct sum of indecomposable $\mathbb{F}_pS_r$-modules, there is a unique summand on which κ_t has non-zero action.
   (b) (2 pts) Furthermore, if Y^μ is a Young module for $\mathbb{F}_pS_r$ which has a submodule isomorphic to S^λ then $\lambda \trianglerighteq \mu$.
   (c) (2 pts) Determine whether or not this gives a proof that the various Young modules Y^λ, as λ ranges through partitions of r, are all non-isomorphic.

Solution: (a) We know from Lemma 2.2.3 that for any λ-tableau t^* we have $\{t^*\}\kappa_t = \pm e_t \in S^\lambda$, so $M^\lambda\kappa_t \subseteq S^\lambda$. In any decomposition $M^\lambda = Y_1 \oplus \cdots \oplus Y_d$ into indecomposable summands, one Y_i contains S^λ, so $M^\lambda\kappa_t \subseteq Y_i$. Since $Y_j\kappa_t \subseteq Y_j$ for all j, $Y_j\kappa_t = 0$ if $j \neq i$. This proves the statement.

(b) This is false. For $\mathbb{F}_2S_2$ we have $S^{[2]} \cong S^{[1^2]} \cong \mathbb{F}_2$, and so $Y^{[2]} = S^{[2]}$ has $S^{[1^2]}$ as a submodule, but $[1^2] \not\trianglerighteq [2]$.

(c) My hope was that the action of κ_t would allow us to distinguish the different Y^λ. Maybe this can be done, but not by the sort of consideration in part (b).

5. In this question, tableaux may have repeated entries. Let λ be a partition of r, and let μ be any sequence of non-negative integers, whose sum is r. We say that a λ-tableau T has type μ if, for every i, the number i occurs μ_i times in T. For example, $\begin{matrix} 2 & 2 & 1 & 1 \\ 1 & & & \end{matrix}$

is a $[4,1]$-tableau of type $[3,2]$. We will number the positions in T according to some tableau with distinct entries, such as

$$t = \begin{matrix} 1 & 2 & 3 & 4 \\ 5 & & & \end{matrix},$$

but it could have been some other such tableau.

(a) (2 pts) Show that the set of λ-tableaux of type μ is in bijection with the set of μ-tabloids.

We now let S_r act on the λ-tableaux of type μ by permuting the positions of the entries. Thus if $T = \begin{matrix} 2 & 2 & 1 & 1 \\ 1 & & & \end{matrix}$ then $T(1,5) = \begin{matrix} 1 & 2 & 1 & 1 \\ 2 & & & \end{matrix}$ and $T(1,5,2) = \begin{matrix} 2 & 1 & 1 & 1 \\ 2 & & & \end{matrix}$ since $(1,5,2) = (1,5)(1,2)$. We say that T_1 and T_2 are row equivalent if $T_2 = T_1\pi$ for some permutation in the row stabilizer of the λ-tableau t.

(b) (2 pts) Show that the row equivalence classes of λ-tableaux of type μ are in bijection with the double cosets $S_\mu \backslash S_r / S_\lambda$.

(c) (2 pts) Show that for each λ-tableau T of type μ there is a RS_r-module homomorphism $\theta_T : M^\lambda \to M^\mu$ such that $\theta_T(\{t\}) = \sum\{T_i \mid T_i \text{ is row equivalent to } T\}$. Thus, in the above example,

$$\theta_T(\{t\}) = \begin{matrix} 2 & 2 & 1 & 1 \\ 1 & & & \end{matrix} + \begin{matrix} 2 & 1 & 2 & 1 \\ 1 & & & \end{matrix} + \begin{matrix} 2 & 1 & 1 & 2 \\ 1 & & & \end{matrix}$$
$$+ \begin{matrix} 1 & 2 & 2 & 1 \\ 1 & & & \end{matrix} + \begin{matrix} 1 & 2 & 1 & 2 \\ 1 & & & \end{matrix} + \begin{matrix} 1 & 1 & 2 & 2 \\ 1 & & & \end{matrix}.$$

(d) (2 pts) Show that, as T ranges over the row equivalence classes of λ-tableaux of type μ the homomorphisms θ_T give a basis for $\mathrm{Hom}_{RS_r}(M^\lambda, M^\mu)$.

Solution: (a) The bijection sends T to the μ-tabloid where row i consists of the positions where the symbol i appears.

(b) This bijection is S_r-equivariant. We see that S_r acts transitively on the λ-tableaux of type μ and the stabilizer of one of them is S_μ, so this S_r-set is $S_\mu \backslash S_r$. The orbits under S_λ are the row equivalence classes, so these biject with $S_\mu \backslash S_r / S_\lambda$.

(c) The λ-tabloid $\{t\}$ bijects with the coset $S_\lambda \cdot 1$ in $S_\lambda \backslash S_r$. In class it was shown that for each coset $S_\mu g \in S_\mu \backslash S_r$, corresponding to T say, there is a homomorphism $M^\lambda \to M^\mu$ determined by $S_\lambda \mapsto$ sum of the S_λ-orbit of $S_\mu g$ in $R[S_\mu \backslash S_r]$. This translates to the expression given in terms of row equivalence.

(d) We have seen in class, with the language of cosets, that this gives a basis for $\mathrm{Hom}_{RS_r}(M^\lambda, M^\mu)$.

6. In this question you may assume that there is a decomposition of the group algebra $\mathbb{F}_2 S_3 \cong Y^{[1^3]} \oplus Y^{[2,1]} \oplus Y^{[2,1]}$ and that $Y^{[1^3]}$ has dimension 2, and has a unique $\mathbb{F}_2 S_3$-submodule of dimension 1. Let $E = \mathbb{F}_2^3$ be a 3-dimensional space over $\mathbb{F}_2$.

(a) (2 pts) Express $E^{\otimes 3}$ as a direct sum of modules M^λ, determining the multiplicity of each M^λ summand.

(b) (2 pts) Make a table with rows and columns indexed by the partitions of 3, whose λ, μ entry is the number of double cosets $|S_\lambda \backslash S_3 / S_\mu|$.

(c) (2 pts) Compute the dimension of $S_{\mathbb{F}_2}(3,3)$.

(d) (2 pts) Compute the dimensions of the simple modules for $S_{\mathbb{F}_2}(3,3)$.

(e) (2 pts) Compute a list of the composition factors of each indecomposable projective $S_{\mathbb{F}_2}(3,3)$-module, assuming the projective has the form $S_{\mathbb{F}_2}(3,3)e$ for some idempotent e.

(f) (2 pts) Show that, as $S_{\mathbb{F}_2}(3,3)$-modules, the symmetric tensors $ST^3(E)$ is indecomposable, but that $E^{\otimes 3}$ is the direct sum of three indecomposable submodules, and find their dimensions.

Solution: (a) $E^{\otimes 3} \cong (M^{[3]})^3 \oplus (M^{[2,1]})^6 \oplus (M^{[1^3]})$ on considering the orbits of basic tensors of shapes $e_1 \otimes e_1 \otimes e_1$, $e_1 \otimes e_1 \otimes e_2$ and $e_1 \otimes e_2 \otimes e_3$.

(b) The table for the partitions $[3], [2,1], [1^3]$ is $\begin{matrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 1 & 3 & 6 \end{matrix}$.

(c) The numbers of double cosets in the table give the dimension of homomorphisms between the M^λ and so the dimension of $S_{\mathbb{F}_2}(3,3)$ is the inner product of $(3,6,1)$ with itself with respect to this matrix, namely 165.

(d) From the decompositions $M^{[3]} = Y^{[3]}$, $M^{[2,1]} = Y^{[3]} \oplus Y^{[2,1]}$ and $M^{[1^3]} = Y^{[1^3]} \oplus Y^{[2,1]} \oplus Y^{[2,1]}$ we get $E^{\otimes 3} \cong (Y^{[3]})^9 \oplus (Y^{[2,1]})^8 \oplus (Y^{[1^3]})$. Each sum of Y^λ for a given λ has endomorphism ring with semisimple quotient a matrix ring of degree the multiplicity of the summand Y^λ, so these multiplicities are the dimensions of the simples: 9, 8 and 1.

(e) We find that $Y^{[2,1]}$ is simple and that $Y^{[3]}$ is the trivial module, and $Y^{[1^2]}$ is described in the question, with two trivial composition factors. Thus, labelling the simples α, β, γ, the projective covers are P_α: uniserial with composition factors γ, α; $P_\beta = \beta$ is simple; P_γ: uniserial with composition factors γ, α, γ.

(f) As with $S_{\mathbb{F}_2}(2,2)$ we find that $ST^3(E) = P_\alpha$ is uniserial. Also $E^{\otimes 3} = Y^{[1^3]\natural} = P_\gamma \oplus P_\beta \oplus P_\beta$ of dimensions $(1+9+1)+8+8 = 11+8+8 (= 27)$.