1. (8 points) Give a proof of the following result by following the suggested steps.

THEOREM. *Let $E \supset F$ be a field extension of finite degree and let A be an F-algebra. Let U and V be A-modules. Then*

$$E \otimes_F \mathrm{Hom}_A(U, V) \cong \mathrm{Hom}_{E\otimes_F A}(E \otimes_F U, E \otimes_F V)$$

via an isomorphism $\lambda \otimes_F f \mapsto (\mu \otimes_F u \mapsto \lambda\mu \otimes_F f(u))$.

(a) Verify that there is indeed a homomorphism as indicated.
Solution: We verify that the specification $\mu \otimes_F u \mapsto \lambda\mu \otimes_F f(u)$ is balanced for F. This is because if $x \in F$ then $\mu x \otimes_F u \mapsto \lambda\mu x \otimes_F f(u) = \lambda\mu \otimes_F xf(u) = \lambda\mu \otimes_F f(xu)$ and this is what $\mu \otimes_F xu$ is sent to. We also check that the assignment on $\lambda \otimes_F f$ is F-balanced by showing similarly that $\lambda x \otimes_F f$ is sent to the same mapping as $\lambda \otimes_F xf$.

(b) Let $x_1, \ldots, x_n$ be a basis for E as an F-vector space. Show that for any F-vector space M, each element of $E \otimes_F M$ can be written uniquely in the form $\sum_{i=1}^n x_i \otimes_F m_i$ with $m_i \in M$.
Solution: Each element of $E \otimes_F M$ can be written in the form $\sum_{i=1}^n \lambda_i x_i \otimes_F u_i$ with $u_i \in M$ and $\lambda_i \in F$. Because E is free as an F-module with the given basis, each term is in the sum has a unique value. Since $\lambda_i x_i \otimes_F u_i = x_i \otimes_F \lambda_i u_i$ and $x_i \otimes_F M \cong M$, putting $m_i = \lambda_i u_i$ we obtain a unique expression for this term as $x_i \otimes_F m_i$.

(c) Show that if an element $\sum_{i=1}^n x_i \otimes f_i \in E \otimes_F \mathrm{Hom}_A(U, V)$ maps to 0 then $\sum_{i=1}^n x_i \otimes f_i(u) = 0$ for all $u \in U$. Deduce that the homomorphism is injective.
Solution: If $\sum_{i=1}^n x_i \otimes f_i$ maps to the zero mapping then the effect of the image map on $1 \otimes u$ is $\sum_{i=1}^n x_i \otimes f_i(u)$, and this is zero, for all u. By the uniqueness from (b) we deduce that $x_i \otimes f_i(u) = 0$ always, which implies $x_i \otimes f_i = 0$ and hence that $\sum_{i=1}^n x_i \otimes f_i = 0$. Thus the homomorphism is injective.

(d) Show that the homomorphism is surjective as follows: given an $E \otimes_F A$-module homomorphism $g : E \otimes_F U \to E \otimes_F V$, write $g(1 \otimes_F u) = \sum_{i=1}^n x_i \otimes f_i(u)$ for some elements $f_i(u) \in V$. Show that this defines A-module homomorphisms $f_i : U \to V$. Show that g is the image of $\sum_{i=1}^n x_i \otimes f_i$.
Solution: If $a \in A$ we have that $\sum_{i=1}^n x_i \otimes f_i(au) = g(1 \otimes_F au) = (1 \otimes_F a)g(1 \otimes_F u) = (1 \otimes_F a) \sum_{i=1}^n x_i \otimes f_i(u) = \sum_{i=1}^n x_i \otimes af_i(u)$. By uniqueness of expression we deduce $f_i(au) = af_i(u)$ always and f_i is an A-module homomorphism. Now the image of $\sum_{i=1}^n x_i \otimes f_i$ equals g on elements $1 \otimes_F u$, and hence equals g since g is E-linear.

2. (5 points) The antiautomorphism of $S_F(n, r)$ used in defining the dual of a representation of the Schur algebra was defined as sending an endomorphism of $E^{\otimes r}$ to its transpose with respect to the standard bilinear form on $E^{\otimes r}$. Compute the effect of

this antiautomorphism on the basis elements $\xi_{\mathbf{i},\mathbf{j}}$ of $S_F(n,r)$ constructed as the duals of the monomial functions $c_{\mathbf{i},\mathbf{j}}$.

Solution: We have seen in class that $\xi_{\mathbf{a},\mathbf{b}}(e_{\mathbf{b}}) = \sum_{\mathbf{k}\in\mathbf{a}\cdot\mathrm{Stab}_{S_r}(\mathbf{b})} e_{\mathbf{k}}$. This means that the matrix of $\xi_{\mathbf{a},\mathbf{b}}$ has a 1 in every position $(\mathbf{a}\pi, \mathbf{b}\pi)$. Applying the antiautomorphism we get an element that acts by the transpose matrix, with a 1 in every position $(\mathbf{b}\pi, \mathbf{a}\pi)$ and this is the matrix of $\xi_{\mathbf{b},\mathbf{a}}$. Thus $\xi_{\mathbf{i},\mathbf{j}}$ is exchanged with $\xi_{\mathbf{j},\mathbf{i}}$.

3. (5 points) For any finite dimensional representation V of a group G we can construct another representation V^* whose representation space is $\mathrm{Hom}_F(V,F)$ and where $g \in G$ acts on a linear map $f : V \to F$ to give ${}^g f$, where ${}^g f(v) = f(g^{?1}v)$. Suppose that F is infinite and V is a polynomial representation of $GL_n(F)$. Show that V^* is polynomial if and only if $GL_n(F)$ acts trivially on V .

Solution: If $g = \mathrm{diag}(t_1,\ldots,t_n)$ then it acts on each weight space V^α as $t_1^{\alpha_1}\cdots t_n^{\alpha_n}$. On V^* the $-\alpha$ weight space is thus nonzero, so that if V^* is polynomial then both α and$-\alpha$ must be non-negative, and hence zero. Thus $V = V^0$ is the zero weight space and the diagonal subgroup acts trivially on V. From this it follows that each diagonalizable element acts trivially on V. Since these elements are dense in $GL_n(F)$, the whole group must act trivially on V. Thus if V is polynomial it must be trivial. Conversely, if V has trivial action then so does V^* so that V^* is polynomial.

4. (5 points) Show that the simple $S_F(n,r)$-modules are self-dual.

Solution: The formal character of a representation and its dual V° are always the same. Since the simple modules are determined by their formal characters, they are self dual. This is because the antiautomorphism of $S_F(n,r)$ fixes ξ_α and so the α-weight spaces of V and V° pair in a non-degenerate fashion, and hence have the same dimension.

5. (5 points) In the situation where we have an algebra B containing an idempotent e and a Schur functor f : B-mod $\to$ eBe-mod, show that the left adjoint and the right adjoint functors of f need not be naturally isomorphic. The left adjoint is $W \mapsto Be \otimes_{eBe} W$ and the right adjoint is $W \mapsto \mathrm{Hom}_{eBe}(eB, W)$.

Solution: Let $A = FS_2$ where $F = \mathbb{F}_2$. We have seen that the regular representation is an indecomposable module with structure $A = \begin{smallmatrix} F \\ F \end{smallmatrix}$ and the Schur algebra $B = S_F(2,2) = \mathrm{End}_A(F \oplus A)$ has dimension 3. Let $e \in B$ be projection onto the second summand. We have seen $eBe \cong A$, and that e and $1-e$ are primitive orthogonal idempotents corresponding to simple B-modules β and α, so that $e\beta \neq 0$ and $e\alpha = 0$, $(1-e)\beta = 0$ and $(1-e)\alpha \neq 0$. Let W be the simple A-module F. If S is a simple B-module then $\mathrm{Hom}_B(Be \otimes_{eBe} W, S) \cong \mathrm{Hom}_{eBe}(W, \mathrm{Hom}_B(Be, S))$. We have also calculated that the projective B-module Be is uniserial with top composition factor β, so the latter group is non-zero when $S \cong \beta$, zero when $S = \alpha$. As a right A-module, $Be \cong F \oplus A$, so $Be \otimes_{eBe} W$ has dimension 2. It follows that $Be \otimes_{eBe} W$ is uniserial with top composition factor β, bottom composition factor α. By a similar argument, $\mathrm{Hom}_B(S, \mathrm{Hom}_A(eB, W)) \cong \mathrm{Hom}_A(eB \otimes_B S, W)$ is nonzero if $S = \beta$, zero if $S = \alpha$. This shows that $Be \otimes_{eBe} W$ is not isomorphic to $\mathrm{Hom}_{eBe}(eB, W)$.