d.
$$\begin{array}{cc}
 & \begin{bmatrix} 1 & 3 & -1 & 4 \\ 1 & 2 & 1 & 2 \\ 3 & 7 & 1 & 9 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 3 & -1 & 4 \\ 0 & -1 & 2 & -2 \\ 0 & -2 & 4 & -3 \end{bmatrix} \\
\begin{bmatrix} 1 & 3 & 0 \\ 0 & -1 & 0 \\ 0 & -2 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 5 & -2 \\ 0 & 1 & -2 & 2 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & 5 & 0 \\ 0 & 1 & -2 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{array}.$$

e.
$$\begin{array}{cc}
 & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 2 & -3 & 3 & 3 \\ 1 & -4 & 2 & 2 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & -5 & 1 & 1 \\ 0 & -5 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & 0 & 0 \\ 0 & -\frac{1}{5} & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & -5 & 1 & 1 \end{bmatrix} \\
\begin{bmatrix} 1 & -1 & 0 \\ 0 & 1 & 0 \\ 0 & 5 & 1 \end{bmatrix} & \begin{bmatrix} 1 & 0 & \frac{6}{5} & \frac{6}{5} \\ 0 & 1 & -\frac{1}{5} & -\frac{1}{5} \\ 0 & 0 & 0 & 0 \end{bmatrix}
\end{array}.$$

2.4.1 The only way you can write

$$\begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix} = a_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + \cdots + a_k \begin{bmatrix} 0 \\ \vdots \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} a_1 \\ a_2 \\ \vdots \\ a_k \end{bmatrix},$$

is if $a_1 = a_2 = \cdots = a_k = 0$.

2.4.2 a. The vectors do form a basis for $\mathbb{R}^3$, since they are three linearly independent vectors: the matrix $\begin{bmatrix} 1 & -2 & -1 \\ 2 & 1 & 1 \\ 3 & 2 & -1 \end{bmatrix}$ row reduces to the identity. The basis is not orthogonal; for example, $\vec{\mathbf{w}}_1 \cdot \vec{\mathbf{w}}_2 = 6 \neq 0$.

b. It is in the span of

$$\begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}, \text{ but not in the span of } \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}, \begin{bmatrix} 5 \\ 1 \\ 4.5 \end{bmatrix}.$$

The matrix formed using those three vectors as the first three columns, and $\begin{bmatrix} 4 \\ 1 \\ 2 \end{bmatrix}$ as the fourth column, row reduces to $\begin{bmatrix} 1 & 0 & 1/2 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$.

2.4.3 To make the basis orthonormal, each vector needs to be normalized to give it length 1. This is done by dividing each vector by its length (see equation 1.4.6). So the orthonormal basis is $\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}$. These vectors form a basis of $\mathbb{R}^2$ because they are two linearly independent vectors in $\mathbb{R}^2$; they are orthogonal because

$$\begin{bmatrix} 1/\sqrt{2} \\ 1/\sqrt{2} \end{bmatrix} \cdot \begin{bmatrix} 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix} = 0. \tag{1}$$

2.4.4 a. By row operations, we can bring the matrix

$$\begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & \alpha \end{bmatrix} \quad \text{to} \quad \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & \alpha - 1 \end{bmatrix}.$$

Therefore, when $\alpha \neq 1$, the vectors are linearly independent.

b. If $\alpha = 1$, the three vectors all lie in the plane of equation $x - y + z = 0$.

2.4.5 To show that $\operatorname{Sp}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$ is a subspace of $\mathbb{R}^n$, we need to show that it is closed under addition and under multiplication by scalars. This follows from the computations

$$\begin{aligned} &c(a_1\vec{\mathbf{v}}_1 + \cdots + a_k\vec{\mathbf{v}}_k) = ca_1\vec{\mathbf{v}}_1 + \cdots + ca_k\vec{\mathbf{v}}_k; \\ &(a_1\vec{\mathbf{v}}_1 + \cdots + a_k\vec{\mathbf{v}}_k) + (b_1\vec{\mathbf{v}}_1 + \cdots + b_k\vec{\mathbf{v}}_k) \\ &\qquad\qquad\qquad = (a_1 + b_1)\vec{\mathbf{v}}_1 + \cdots + (a_k + b_k)\vec{\mathbf{v}}_k. \end{aligned}$$

To see that it is the smallest subspace that contains the $\vec{\mathbf{v}}_i$, note that any subspace that contains the $\vec{\mathbf{v}}_i$ must contain their linear combinations, hence the smallest such subspace is $\operatorname{Sp}(\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k)$.

2.4.6

2.4.7 Let A be an $n \times n$ matrix. The product $A^\top A$ is then

$$\underbrace{\begin{bmatrix} \dots & \vec{\mathbf{a}}_1^\top & \dots \\ \dots & \vec{\mathbf{a}}_2^\top & \dots \\ \dots & \dots & \dots \\ \dots & \vec{\mathbf{a}}_n^\top & \dots \end{bmatrix}}_{A^\top} \quad \begin{matrix} \overbrace{\begin{bmatrix} \vdots & \vdots & \dots & \vdots \\ \\ \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_n \\ \vdots & \vdots & \dots & \vdots \end{bmatrix}}^{A} \\ \underbrace{\begin{bmatrix} |\vec{\mathbf{a}}_1|^2 & \vec{\mathbf{a}}_1 \cdot \vec{\mathbf{a}}_2 & \dots & \vec{\mathbf{a}}_1 \cdot \vec{\mathbf{a}}_n \\ \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_1 & |\vec{\mathbf{a}}_2|^2 & \dots & \vec{\mathbf{a}}_2 \cdot \vec{\mathbf{a}}_n \\ \vdots & \vdots & \ddots & \dots \\ \vec{\mathbf{a}}_n \cdot \vec{\mathbf{a}}_1 & \vec{\mathbf{a}}_n \cdot \vec{\mathbf{a}}_2 & \dots & |\vec{\mathbf{a}}_n|^2 \end{bmatrix}}_{A^\top A} \end{matrix}.$$

An orthogonal $n \times n$ matrix is a matrix whose columns form an orthonormal basis of $\mathbb{R}^n$.

The diagonal entries are given by the length squared of the columns of A, since $\vec{\mathbf{a}}_i^\top \vec{\mathbf{a}}_i = \vec{\mathbf{a}}_i \cdot \vec{\mathbf{a}}_i = |\vec{\mathbf{a}}_i|^2$. All other entries are dot products of two different columns of A. If $A^\top A = I$, so that all entries not on the diagonal are 0, while those on the diagonal are 1, then the columns of A are orthogonal and have length 1. Thus they form an orthonormal basis of $\mathbb{R}^n$, and A is said to be orthogonal.

Similarly, if A is orthogonal, then the length of each of its column vectors is 1, so that $A^\top A$ has 1's on the diagonal, and the dot product of two non-identical columns is 0, giving 0 for all other entries of $A^\top A$.

2.4.8

2.4.9 To see that condition 2 implies condition 3, first note that $2 \implies 3$ is logically equivalent to (not 3) $\implies$ (not 2). Now suppose $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ is a *linearly dependent* set spanning V, so by definition 2.4.10, there exists a nontrivial solution to

$$a_1\vec{\mathbf{v}}_1 + \cdots + a_k\vec{\mathbf{v}}_k = \mathbf{0}.$$

Without loss of generality, we may assume that a_k is nonzero (if it isn't, renumber the vectors so that a_k is nonzero). Using the above relation, we can solve for $\vec{\mathbf{v}}_k$ in terms of the other vectors:

$$\vec{\mathbf{v}}_k = -\left(\frac{a_1}{a_k}\vec{\mathbf{v}}_1 + \cdots + \frac{a_{k-1}}{a_k}\vec{\mathbf{v}}_{k-1}\right).$$

This implies that $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ cannot be a minimal spanning set, because if we were to drop $\vec{\mathbf{v}}_k$ we could still form all the linear combinations as before. So (not 3) $\implies$ (not 2), and we are finished.

To show that $3 \implies 1$:

The vectors $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$ span V, so for any vector $\vec{\mathbf{w}} \in V$, there exist some numbers $a_1, \dots, a_n$ such that

$$a_1\vec{\mathbf{v}}_1 + \cdots + a_n\vec{\mathbf{v}}_n = \vec{\mathbf{w}}.$$

Thus, if we add this vector to $\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k$, we will have a linearly dependent set because

$$a_1\vec{\mathbf{v}}_1 + \cdots + a_n\vec{\mathbf{v}}_n - \vec{\mathbf{w}} = \mathbf{0}$$

is a nontrivial linear combination of the vectors that equals $\mathbf{0}$. Since $\vec{\mathbf{w}}$ can be any vector in V, $\{\vec{\mathbf{v}}_1, \dots, \vec{\mathbf{v}}_k\}$ is a maximal linearly independent set.

2.4.10 Let u be the coefficient of $\vec{\mathbf{v}}_1$ and v the coefficient of $\vec{\mathbf{v}}_2$. The equations are then $u + v = x$ and $u + 3v = y$, which could also be written as the matrix multiplication

$$\begin{bmatrix} 1 & 1 \\ 1 & 3 \end{bmatrix}\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

This can be solved for u and v, to give

$$\begin{aligned} u &= (3x - y)/2 \\ v &= (y - x)/2. \end{aligned}$$

Thus we have

$$\begin{bmatrix} 3 \\ -5 \end{bmatrix} = \frac{3 \cdot 3 + 5}{2}\vec{\mathbf{v}}_1 + \frac{-5 - 3}{2}\vec{\mathbf{v}}_2 = 7\vec{\mathbf{v}}_1 - 4\vec{\mathbf{v}}_2 = 7\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

2.4.11 a. For any n, we have $n+1$ linear equations for the $n+1$ unknowns $a_{0,n}, a_{1,n}, \dots, a_{n,n}$, which say

$$a_{0,n}\left(\frac{0}{n}\right)^k + a_{1,n}\left(\frac{1}{n}\right)^k + a_{2,n}\left(\frac{2}{n}\right)^k + \cdots + a_{n,n}\left(\frac{n}{n}\right)^k = \int_0^1 x^k\,dx = \frac{1}{k+1},$$

one for each $k = 0, 1, \dots, n$.

These systems of linear equations are:

- When $n = 1$

$$a_{0,1}1 + a_{1,1}1 = 1$$
$$a_{0,1}0 + a_{1,1}1 = 1/2$$

- When $n = 2$

$$a_{0,2}1 + a_{1,2}1 + a_{2,2}1 = 1$$
$$a_{0,2}0 + a_{1,2}(1/2) + a_{2,2}1 = 1/2$$
$$a_{0,2}0 + a_{1,2}(1/4) + a_{2,2}1 = 1/3$$

- When $n = 3$

$$a_{0,3}1 + a_{1,3}1 + a_{2,3}1 + a_{3,3}1 = 1$$
$$a_{0,3}0 + a_{1,3}(1/3) + a_{2,3}(2/3) + a_{3,3}1 = 1/2$$
$$a_{0,3}0 + a_{1,3}(1/9) + a_{2,3}(4/9) + a_{3,3}1 = 1/3$$
$$a_{0,3}0 + a_{1,3}(1/27) + a_{2,3}(8/27) + a_{3,3}1 = 1/4.$$

The system of equations for $n = 3$ could be written as the augmented matrix $[A|\vec{\mathbf{b}}]$:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1/3 & 2/3 & 1 & 1/2 \\ 0 & 1/9 & 4/9 & 1 & 1/3 \\ 0 & 1/27 & 8/27 & 1 & 1/4 \end{bmatrix}.$$

b. These wouldn't be too bad to solve by hand (although already the last would be distinctly unpleasant). We wrote a little MATLAB m-file to do it systematically:

```
function [N,b,c] = EqSp(n)

N = zeros(n+1); % make an n+1 × n+1 matrix of zeros
c=linspace(1,n+1,n+1); % make a place holder for the right side
for i=1:n+1
    for j=1:n+1
    N(i,j)= ((j-1)/n)^(i-1); % put the right coefficients in the matrix
end
c(i)=1/c(i); % put the right entries in the right side
end
b=c'; % our c was a row vector, take its transpose
c=N% this solves the system of linear equations
```

If you write and save this file as 'EqSp.m', and then type

[A,b,c]=EqSp(5), for the case when $n = 5$,

you will get

$$A = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1/5 & 2/5 & 3/5 & 4/5 & 1 \\ 0 & 1/25 & 4/25 & 9/25 & 16/25 & 1 \\ 0 & 1/125 & 8/125 & 27/125 & 64/125 & 1 \\ 0 & 1/625 & 16/625 & 81/625 & 256/625 & 1 \\ 0 & 1/3125 & 32/3125 & 243/3125 & 541/1651 & 1 \end{bmatrix}, \quad b = \begin{bmatrix} 1 \\ 1/2 \\ 1/3 \\ 1/4 \\ 1/5 \\ 1/6 \end{bmatrix}, \quad c = \begin{bmatrix} 19/288 \\ 25/96 \\ 25/144 \\ 25/144 \\ 25/96 \\ 19/288 \end{bmatrix}.$$

This corresponds to the equation $Ac = b$, where the matrix A is the matrix of coefficients for $n = 5$, and the vector c is the desired set of coefficients – the solutions when $n = 5$.

When $n = 1$, 2,3, the coefficients – i.e., the solutions to the systems of equations in part a – are

For instance, for $n = 2$, we have

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix} = \begin{bmatrix} 1 \\ 1/2 \end{bmatrix}.$$

$$\begin{bmatrix} 1/2 \\ 1/2 \end{bmatrix}, \quad \begin{bmatrix} 1/6 \\ 2/3 \\ 1/6 \end{bmatrix}, \quad \begin{bmatrix} 1/8 \\ 3/8 \\ 3/8 \\ 1/8 \end{bmatrix}.$$

The approximations to $\int_0^1 \frac{dx}{1+x} = \log 2 = 0.69314718055995\ldots$ obtained with these coefficients are .75 for $n = 1$, $\frac{25}{36} = .6944\ldots$ for $n = 2$, and $\frac{111}{160} = .69375$ for $n = 3$.

c. If you compute

$$\sum_{i=0}^{5} a_{i,5} \frac{1}{(i/5)+1} \approx \int_0^1 \frac{dx}{1+x} = \log 2 = 0.69314718055995\ldots$$

you will find 0.69316302910053, which is a pretty good approximation for a Riemann sum with six terms. For instance, the midpoint Riemann sum gives

$$\frac{1}{5} \sum_{i=1}^{5} \frac{1}{((2i-1)/10)} \approx 0.69190788571594,$$

which is a much worse approximation. But this scheme runs into trouble. All the coefficients are positive up to $n = 7$, but for $n = 8$ they are

$$\begin{bmatrix} 248/7109 \\ 578/2783 \\ -111/3391 \\ 97/262 \\ -454/2835 \\ 97/262 \\ -111/3391 \\ 578/2783 \\ 248/7109 \end{bmatrix} \approx \begin{bmatrix} 0.0349 \\ 0.2077 \\ -0.0327 \\ 0.3702 \\ -0.1601 \\ 0.3702 \\ -0.0327 \\ 0.2077 \\ 0.0349 \end{bmatrix}$$

and the approximation scheme starts depending on cancellations. This is much worse when $n = 20$, where the coefficients are as shown in the margin.

$$\begin{bmatrix} 0.0118 \\ 0.1141 \\ -0.2362 \\ 1.2044 \\ -3.7636 \\ 10.3135 \\ -22.6521 \\ 41.7176 \\ -63.9006 \\ 82.5706 \\ -89.7629 \\ 82.5829 \\ -63.9189 \\ 41.7345 \\ -22.6633 \\ 10.3191 \\ -3.7656 \\ 1.2050 \\ -0.2363 \\ 0.1141 \\ 0.0118 \end{bmatrix}.$$

Coefficients when $n = 20$.

Despite these bad sign variations, the Riemann sum works pretty well: the approximation to the integral above gives 0.69314718055995, which is ln 2 to the precision of the machine.

2.4.12 a. If we identify $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $\begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix}$, the matrices I, A_t, A_t^2, A_t^3 become

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ t \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 4 \\ 4t \\ 0 \\ 4 \end{bmatrix}, \quad \begin{bmatrix} 8 \\ 12t \\ 0 \\ 8 \end{bmatrix}.$$

The matrix with these columns can be brought by row operations to

$$\begin{bmatrix} 1 & 2 & 4 & 8 \\ 0 & t & 4t & 12t \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Thus we see that if $t \neq 0$, the subspace V_t has dimension 2, and if $t = 0$, then V_t has dimension 1.

b. We need to show that the set W_t is closed under addition and multiplication by scalars. If $B_1A = AB_1$ and $B_2A = AB_2$, adding the equations gives

$$(B_1 + B_2)A = B_1A + B_2A = AB_1 + AB_2 = A(B_1 + B_2).$$

Similarly, if $BA = AB$, then $(aB_1)A = aB_1A = aAB_1 = A(aB_1)$.

The multiplications

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 2a & ta + 2b \\ 2c & tc + 2d \end{bmatrix} \text{ and } \begin{bmatrix} 2 & t \\ 0 & 2 \end{bmatrix}\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 2a + tc & 2b + td \\ 2c & 2d \end{bmatrix}$$

give the equations

$$2a = 2a + tc,\ ta + 2b = 2b + td,\ 2c = 2c,\ tc + 2d = 2d$$

for the subspace W_t. If $t = 0$, all the equations are automatically satisfied, so $W_0 = \text{Mat}\,(2, 2)$. But if $t \neq 0$, these equations boil down to $a = d, c = 0$. So W_t has dimension 2 if $t \neq 0$.

c. Since $A\,A^k = A^{k+1} = A^k\,A$, we see that the matrices that span V_t are all in W_t, so $V_t \subset W_t$. If $t = 0$, they are different, since V_t has dimension 1 and W_t has dimension 4. But if $t \neq 0$, they both have dimension 2, so they are equal.

2.4.13 In the process of row reducing $A = \begin{bmatrix} 1 & a & a & a \\ 1 & 1 & a & a \\ 1 & 1 & 1 & a \\ 1 & 1 & 1 & 1 \end{bmatrix}$, you will come to the matrix

$$\begin{bmatrix} 1 & a & a & a \\ 0 & 1-a & 0 & 0 \\ 0 & 1-a & 1-a & 0 \\ 0 & 1-a & 1-a & 1-a \end{bmatrix}.$$

If $a = 1$, the matrix will not row reduce to the identity, because you can't choose a pivotal 1 in the second column, so one necessary condition for A to be invertible is that $a \neq 1$. Let us suppose that this is the case, we can now row reduce two steps further to find

$$\begin{bmatrix} 1 & 0 & a & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1-a & 0 \\ 0 & 0 & 1-a & 1-a \end{bmatrix}, \quad \text{and then} \quad \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1-a \end{bmatrix}$$

The next step row reduces the matrix to the identity, so the matrix is invertible if and only if $a \neq 1$.

2.5.1 a. The vectors $\vec{\mathbf{v}}_1$ and $\vec{\mathbf{v}}_3$ are in the kernel of A, since $A\vec{\mathbf{v}}_1 = \mathbf{0}$ and $A\vec{\mathbf{v}}_3 = \mathbf{0}$. But $\vec{\mathbf{v}}_2$ is not, since $A\vec{\mathbf{v}}_2 = \begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$. The vector $\begin{bmatrix} 2 \\ 3 \\ 3 \end{bmatrix}$ is in the image of A.

b. The matrix T represents a transformation from $\mathbb{R}^5$ to $\mathbb{R}^3$; it takes a vector in $\mathbb{R}^5$ and gives a vector in $\mathbb{R}^3$. Therefore, $\vec{\mathbf{w}}_4$ has the right height to be in the kernel (although it isn't), and $\vec{\mathbf{w}}_1$ and $\vec{\mathbf{w}}_3$ have the right height to be in its image.

Since the sum of the second and fifth columns of T is $\vec{\mathbf{0}}$, one element of the kernel is $\begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

2.5.2 a. False (unless $n = m$) b. True c. True d. False (unless $n = m$)

e. False (the nullity of T is the dimension of its kernel, which is $n - m$)

f. False (unless $n = m$) g. False (unless $n = m$)

2.5.3 nullity $T = \dim \ker T$ = number of nonpivotal columns of T;

$$\begin{aligned} \text{rank of } T &= \text{dim image } T \\ &= \text{number of linearly independent columns of } T \\ &= \text{number of pivotal columns of } T. \end{aligned}$$

rank $T+$ nullity T = dim domain T

2.5.4 An $n \times m$ matrix A represents a linear function from $\mathbb{R}^m$ to $\mathbb{R}^n$. If $\tilde{A}$ has at least one row containing all 0's, then A has rank $< n$. Indeed, the rank of A is the number of pivotal 1's of $\tilde{A}$, and there is at most one per row.

If $\tilde{A}$ has exactly one row of 0's, then the same argument says that the rank of A is $n - 1$.

2.5.5 By definition 1.1.5 of a subspace, we need to show that the kernel and the image of a linear transformation T are closed under addition and multiplication by scalars. These are straightforward computations, using the linearity of T.

The kernel of T: If $\vec{\mathbf{v}}, \vec{\mathbf{w}} \in \ker T$, i.e., if $T(\vec{\mathbf{v}}) = \vec{\mathbf{0}}$ and $T(\vec{\mathbf{w}}) = \vec{\mathbf{0}}$, then

$$T(\vec{\mathbf{v}} + \vec{\mathbf{w}}) = T(\vec{\mathbf{v}}) + T(\vec{\mathbf{w}}) = \vec{\mathbf{0}} + \vec{\mathbf{0}} = \vec{\mathbf{0}} \quad \text{and} \quad T(a\vec{\mathbf{v}}) = aT(\vec{\mathbf{v}}) = a\vec{\mathbf{0}} = \vec{\mathbf{0}},$$

so $\vec{\mathbf{v}} + \vec{\mathbf{w}} \in \ker T$ and $a\vec{\mathbf{v}} \in \ker T$.

The image of T: If $\vec{\mathbf{v}} = T(\vec{\mathbf{v}}_1)$, $\vec{\mathbf{w}} = T(\vec{\mathbf{w}}_1)$, then

$$\vec{\mathbf{v}} + \vec{\mathbf{w}} = T(\vec{\mathbf{w}}_1) + T(\vec{\mathbf{v}}_1) = T(\vec{\mathbf{w}}_1 + \vec{\mathbf{v}}_1) \quad \text{and} \quad a\vec{\mathbf{v}} = aT(\vec{\mathbf{v}}_1) = T(a\vec{\mathbf{v}}_1),$$

So the image is also closed under addition and multiplication by scalars.

2.5.6 a. If you row reduce $\begin{bmatrix} 1 & 1 & 3 \\ 2 & 2 & 6 \end{bmatrix}$ you get $\begin{bmatrix} 1 & 1 & 3 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the first column $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is a basis of the image (which has dimension 1), and the two vectors

$$\begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix},$$

which are the solutions of $x+y+3z=0$ with respectively $y=1, z=0$ and $y=0, z=1$, form a basis of the kernel.

b. If you row reduce $\begin{bmatrix} 1 & 2 & 3 \\ -1 & 1 & 1 \\ -1 & 4 & 5 \end{bmatrix}$ you get $\begin{bmatrix} 1 & 0 & 1/3 \\ 0 & 1 & 4/3 \\ 0 & 0 & 0 \end{bmatrix}$. Thus the first two columns $\begin{bmatrix} 1 \\ -1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}$ form a basis of the image (which has dimension 2), and the vector $\begin{bmatrix} -1/3 \\ -4/3 \\ 1 \end{bmatrix}$ is a basis of the kernel.

c. The matrix $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \\ 2 & 3 & 4 \end{bmatrix}$ row reduces to $\begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix}$. Again, the first two columns $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ form a basis of the image, and the vector $\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$ forms a basis of the kernel.

2.5.7 a. $n = 3$. The last three columns of the matrix are clearly linearly independent, so the matrix has rank at least 3, and it has rank at most 3 because there can be at most three linearly independent vectors in $\mathbb{R}^3$.

In this case, all triples of vectors are linearly independent except columns 3, 4, and 6.

b. Yes. For example, the first three columns are linearly independent, since the matrix composed of just those columns row reduces to the identity.

c. The 3rd, 4th, and 6th columns are linearly dependent.

d. You cannot choose freely the values of x_1, x_2, x_5. Since the rank of the matrix is 3, three variables must correspond to pivotal (linearly independent) columns. For the variables x_1, x_2, x_5 to be freely chosen, i.e., nonpivotal, x_3, x_4, x_6 would have to correspond to linearly independent columns.