So if you set $\delta = \epsilon$, and $|H| \le \delta$, then equation (2) is satisfied.

c. We will show that the limit does not exist. In this case, we find

$$\begin{aligned}(A+H-A)^{-1}(A+H)^2 - A^2 &= H^{-1}(I^2 + AH + HA + H^2 - I^2)\\ &= H^{-1}(AH + HA + H^2) = A + H^{-1}AH + H^2.\end{aligned}$$

If the limit exists, it must be $2A$: choose $H = \epsilon I$ so that $H^{-1} = \epsilon^{-1}I$; then

$$A + H^{-1}AH + H^2 = 2A + \epsilon I$$

is close to $2A$.

But if you choose $H = \epsilon \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$, you will find that

$$H^{-1}AH = \begin{bmatrix} 1/\epsilon & 0 \\ 0 & -1/\epsilon \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} \epsilon & 0 \\ 0 & \epsilon \end{bmatrix} = \begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix} = -A.$$

So with this H we have

$$A + H^{-1}AH + H^2 = A - A + \epsilon H$$

which is close to the zero matrix.

1.5.24

1.6.1 Let B be a set contained in a ball of radius R centered at a point $\mathbf{a}$. Then it is also contained in a ball of radius $R + |\mathbf{a}|$ centered at the origin; thus it is bounded.

1.6.2 First, remember that compact is equivalent to closed and bounded so if A is not compact then A is unbounded and/or not closed. If A is unbounded then the hint is sufficient. If A is not closed then A has a limit point $\mathbf{a}$ not in A: i.e., there exists a sequence in A that converges in $\mathbb{R}^n$ to a point $\mathbf{a} \notin A$. Use this $\mathbf{a}$ as the $\mathbf{a}$ in the hint.

1.6.3 The polynomial $p(z) = 1 + x^2y^2$ has no roots because 1 plus something positive cannot be 0. This does not contradict the fundamental theorem of algebra because although p is a polynomial in the real variables x and y, it is not a polynomial in the complex variable z: it is a polynomial in z and $\bar{z}$. It is possible to write $p(z) = 1 + x^2y^2$ in terms of z and $\bar{z}$. You can use

$$x = \frac{z + \overline{z}}{2} \quad \text{and} \quad y = \frac{z - \overline{z}}{2i},$$

and find

$$p(z) = 1 + \frac{z^4 - 2|z|^4 + \overline{z}^4}{-16} \tag{1}$$

but you simply cannot get rid of the $\overline{z}$.

1.6.4 If $|z| \ge 4$, then

$$|p(z)| \ge |z|^5 - 4|z|^3 - 3|z| - 3 > |z|^5 - 4|z|^3 - 3|z|^3 - 3|z|^3 = |z|^3(|z|^2 - 10) \ge 6 \cdot 4^3.$$

Since the disk $|z| \leq 4$ is closed and bounded, and since $|p(z)|$ is continuous, the function $|p(z)|$ has a minimum in the disk $|z| \leq 4$ at some point z_0. Since $|p(0)| = 3$, the minimum value is smaller than 3, so $|z_0| \neq 4$, and is the absolute minimum of $|p(z)|$ over all of $\mathbb{C}$. We know that then z_0 is a root of p.

1.6.5 a. Suppose $|z| > 3$. Then

$$|z|^6 - |q(z)| \geq |z|^6 - (4|z|^4 + |z| + 2) \geq |z|^6 - (4|z|^4 + |z|^4 + 2|z|^4) = |z|^4(|z|^2 - 7) \geq (9-7) \cdot 3^4 = 162.$$

How did we come by the number 3? We started the computation, until we got to the expression $|z|^2 - 7$, which we needed to be positive. The number 3 works, and 2 does not; 2.7 works too.

b. Since $p(0) = 2$, but when $|z| > 3$ we have $|p(z)| \geq |z|^6 - |q(z)| \geq 162$, the minimum of $|p|$ on the disc of radius $R_1 = 3$ around the origin must be the absolute minimum of $|p|$. Notice that this minimum must exist, since it is a minimum of the continuous function $|p(z)|$ on the closed and bounded set $|z| \leq 3$ of $\mathbb{C}$.

1.6.6 a. The function xe^{-x} has derivative $(1-x)e^{-x}$ which is negative if $x > 1$. Hence $\sup_{x \in [1,\infty)} xe^{-x} = 1 \cdot e^{-1} = 1/e$. So

$$\sup_{x \in \mathbb{R}} |x|e^{-|x|} = \sup_{x \in [-1,1]} |x|e^{-|x|},$$

and this supremum is achieved, since $|x|e^{-|x|}$ is a continuous function and $[-1,1]$ is compact.

b. The maximum value must occur on $(0,\infty)$, hence at a point where the function is differentiable, and the derivative is 0. This happens only at $x = 1$, so the absolute maximum value is $1/e$.

c. The image of f is certainly contained in $[0, 1/e]$, since the function takes only non-negative values, and it has an absolute maximum value of $1/e$. Given any $y \in [0, 1/e]$, the function $f(x) - y$ is ≤ 0 at 0 and ≥ 0 at 1, so by the intermediate value theorem it must vanish for some $x \in [0,1]$, so every $y \in [0, 1/e]$ is in the image of f.

1.6.7 Consider the function $g(x) = f(x) - mx$. This is a continuous function on the closed and bounded set $[a,b]$, so it has a minimum at some point $c \in [a,b]$. Let us see that $c \neq a$ and $c \neq b$. Since $g'(a) = f'(a) - m < 0$, we have

$$\lim_{h \to 0} \frac{g(a+h) - g(a)}{h} < 0.$$

Let us spell this out: for every $\epsilon > 0$, there exists $\delta > 0$ such that $0 < |h| < \delta$ implies

$$\left| \frac{g(a+h) - g(a)}{h} - g'(a) \right| < \epsilon.$$

Solution 1.6.7: Although our function g is differentiable on a neighborhood of a and b, we cannot apply proposition 1.6.11 if the minimum occurs at one of those points, since c would not be a maximum on a neighborhood of the point.

Choose $\epsilon = |g'(a)|/2$, and find a corresponding $\delta > 0$, and set $h = \delta/2$. Then the inequality

$$\left| \frac{g(a+h) - g(a)}{h} - g'(a) \right| < \frac{|g'(a)|}{2}$$

implies that

$$\frac{g(a+h)-g(a)}{h} < \frac{g'(a)}{2} < 0$$

and since $h > 0$ we have $g(a+h) < g(a)$, so a is not the minimum of g.

Similarly, b is not the minimum:

$$\lim_{h \to 0} \frac{g(b+h)-g(b)}{h} = g'(b) - m > 0.$$

Express this again in terms of ϵ's and δ's, choose $\epsilon = g'(b)/2$, and set $h = -\delta/2$. As above, we have

$$\frac{g(b+h)-g(b)}{h} > \frac{g'(b)}{2} > 0,$$

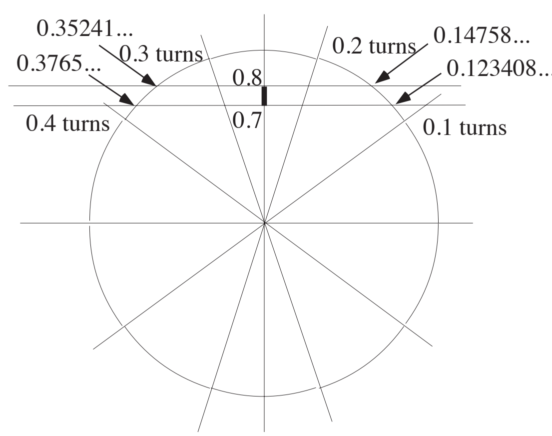
and since $h < 0$, this implies $g(b+h) < g(b)$.

So $c \in (a,b)$, and in particular c in a minimum on (a,b), so $g'(c) = f'(c) - m = 0$ by proposition 1.6.11.

1.6.8 In order for the sequence $\sin 10^n$ to have a subsequence that converges to a limit in $[.7, .8]$, it is necessary that 10^n radians be either in the arc of circle bounded by $\arcsin .7$ and $\arcsin .8$ or in the arc bounded by $(\pi - \arcsin .7)$ and $(\pi - \arcsin .8)$, since these also have sines in the desired interval.

As described in the example, it is easier to think that $10^n/(2\pi)$ turns (as opposed to radians) lies in the same arcs. Since the whole turns don't count, this means that the fractional part of $10^n/(2\pi)$ turns lies in the arcs above, i.e., that the number obtained by moving the decimal point to the right by n positions and discarding the part to the left of it lies in the intervals.

The following picture illustrates where the sine lies, and where the numbers "fractional part of $10^n/(2\pi)$" must lie.



The calculator says

$$\arcsin .7/(2\pi) \approx .123408, \quad \text{and} \quad .5 - \arcsin .7/(2\pi) \approx .3765$$
$$\arcsin .8/(2\pi) \approx .14758, \quad \text{and} \quad .5 - \arcsin .7/(2\pi) \approx .35241,$$

we see that in order for the sequence $\sin 10^n$ to have a subsequence with a limit in $[.7, .8]$, it is necessary that there be infinitely many 1's in the decimal expansion of $1/(2\pi)$, or infinitely many 3's (or both). In fact, we can say more: there must be infinitely many 1's followed by $2, 3$ or 4, or infinitely many 3's followed by $5, 6$ or 7 (or both). Even these are not sufficient conditions; but a sufficient condition would be that there are infinitely many 1 followed by 3, or infinitely many 3's followed by 6.

Remark. According to Maple,

$$\frac{1}{2\pi} = .15915494309189533576888376337251\mathbf{1436}203445964574045$$
$$644874766734405889679763422653509011\mathbf{13}38027662530860\ldots$$

to 100 places. We do see a few such sequences of two digits (three of them if I counted up right). This is about what one would expect for a random sequence of digits, but not really evidence one way or the other for whether there is a limit

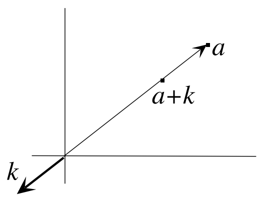


Figure for solution 1.6.9

A first error to avoid is writing "$a + bu^j$ is between 0 and a" as

"$0 < a + bu^j < a$."

Remember that a, b, and u are complex numbers so that writing that sort of inequality doesn't make sense. If we set $k = bu^j$ to simpify notation, then $a + k$ is between 0 and a if $a-(a+k) = k$ is on the same line as a and points in the opposite direction, witih $|k| < |a|$.

The proof given essentially reproves proposition 0.7.7. If you want to use that proposition instead, you could say:

If $a + bu^j$ is between 0 and a, then there exists ρ with $0 < \rho < 1$ such that

$$a + bu^j = \rho a, \text{ i.e., } u^j = \frac{(\rho - 1)a}{b}.$$

This equation has j solutions by proposition 0.7.7, and

$$|u| = (1 - \rho)|a/b| < |a/b|,$$

so we can take $p_0 = |a/b|^{1/j}$.

1.6.9 A first error to avoid is writing "$a + bu^j$ is between 0 and a" as "$0 < a + bu^j < a$." Remember that a, b, and u are complex numbers, so that writing that sort of inequality doesn't make sense. "Between 0 and a" means that if you plot a as a point in $\mathbb{R}^2$ in the usual way (real part of a on the x-axis, imaginary part on the y-axis), then $a + bu^j$ lies on the line connecting the origin and the point a.

For this to happen, bu^j must point in the opposite direction as a, and we must have $|bu^j| < |a|$. Write

$$a = r_1(\cos\omega_1 + i\sin\omega_1)$$
$$b = r_2(\cos\omega_2 + i\sin\omega_2)$$
$$u = p(\cos\theta + i\sin\theta).$$

Then

$$a + bu^j = r_1(\cos\omega_1 + i\sin\omega_1) + r_2 p^j\big(\cos(\omega_2 + j\theta) + i\sin(\omega_2 + j\theta)\big).$$

Then bu^j will point in the opposite direction from a if

$$\omega_2 + j\theta = \omega_1 + \pi + 2k\pi \text{ for some } k, \text{ i.e. }, \theta = \frac{1}{j}(\omega_1 - \omega_2 + \pi + 2k\pi),$$

and we find j distinct such angles by taking $k = 0, 1, \ldots, j - 1$.

The condition $|bu^j| < |a|$ becomes $r_2 p^j < r_1$, so we can take $0 < p < (r_1/r_2)^{1/j} \stackrel{\text{def}}{=} p_0$.

1.6.10

1.6.11 Set $p(x) = x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0$ with k odd. Choose

$$C = \sup\{1, |a_{k-1}|, \ldots, |a_0|\}$$

and set $A = kC + 1$. Then if $x \le -A$ we have

$$\begin{aligned} p(x) &= x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \\ &\le (-A)^k + CA^{k-1} + \cdots + C \le -A^k + kCA^{k-1} \\ &= A^{k-1}(kC - A) = -A^{k-1} \le 0. \end{aligned}$$

Similarly, if $x \ge A$ we have

$$\begin{aligned} p(x) &= x^k + a_{k-1}x^{k-1} + \cdots + a_1 x + a_0 \\ &\ge (A)^k - CA^{k-1} - \cdots - C \ge A^k - kCA^{k-1} \\ &= A^{k-1}(A - kC) = A^{k-1} \ge 0. \end{aligned}$$

Since $p : [-A, A] \to \mathbb{R}$ is a continuous function (corollary 1.5.30) and we have $p(-A) \le 0$ and $p(A) \ge 0$, then by the intermediate value theorem there exists $x_0 \in [-A, A]$ such that $p(x_0) = 0$.

1.7.1 a. $f(a) = 0$, $f'(a) = \cos(a) = 1$ so the tangent is $g(x) = x$.

b. $f(a) = \frac{1}{2}$, $f'(a) = -\sin(a) = -\frac{\sqrt{3}}{2}$ so the tangent is

$$g(x) = -\frac{\sqrt{3}}{2}(x - \frac{\pi}{3}) + \frac{1}{2}.$$

c. $f(a) = 1$, $f'(a) = -\sin(a) = 0$ so the tangent is $g(x) = 1$.

d. $f(a) = 2$, $f'(a) = -\frac{1}{a^2} = -4$ so the tangent is

$$g(x) = -4(x - 1/2) + 2 = -4x + 4.$$

1.7.2 We need to find a such that if the graph of g is the tangent at a, then $g(0) = 0$. Since the tangent is

$$g(x) = e^{-a} - e^{-a}(x - a),$$

we have

$$g(0) = e^{-a} + ae^{-a} = 0,$$

so

$$e^{-a}(1 + a) = 0, \quad \text{which gives} \quad a = -1.$$

1.7.3 a. $f'(x) = \Big(3\sin^2(x^2 + \cos x)\Big)\Big(\cos(x^2 + \cos x)\Big)\Big(2x - \sin x\Big)$

b. $f'(x) = \Big(2\cos((x + \sin x)^2)\Big)\Big(-\sin((x + \sin x)^2)\Big)\Big(2(x + \sin x)\Big)\Big(1 + \cos x\Big)$

c. $f'(x) = \Big((\cos x)^5 + \sin x\Big)\Big(4(\cos x)^3\Big)(-\sin(x)) = (\cos x)^5 - 4(\sin x)^2(\cos x)^3$

d. $f'(x) = 3(x + \sin^4 x)^2(1 + 4\sin^3 x \cos x)$

e. $f'(x) = \dfrac{\sin^3 x(\cos x^2 * 2x)}{2 + \sin(x)} + \dfrac{\sin x^2(3\sin^2 x \cos x)}{2 + \sin(x)} - \dfrac{(\sin x^2 \sin^3 x)(\cos x)}{\big(2 + \sin(x)\big)^2}$

f. $f'(x) = \cos\left(\dfrac{x^3}{\sin x^2}\right)\left(\dfrac{3x^2}{\sin x^2} - \dfrac{(x^3)(\cos x^2 * 2x)}{(\sin x^2)^2}\right)$

1.7.4 a. If $f(x) = |x|^{3/2}$, then

$$f'(0) = \lim_{h\to 0} \frac{|h|^{3/2}}{h} = \lim_{h\to 0} |h|^{1/2} = 0,$$

so the derivative does exist. But

$$f(0+h) - f(0) - hf'(0) = |h|^{3/2}$$

is larger than h^2, since the limit

$$\lim_{h\to 0} \frac{|h|^{3/2}}{h^2} = \lim_{h\to 0} |h|^{-1/2}$$

is infinite.

b. If $f(x) = x \ln |x|$, then the limit

$$f'(0) = \lim_{h\to 0} \frac{h \ln |h|}{h} = \lim_{h\to 0} \ln |h|,$$

is infinite, and the derivative does not exist.

c. If $f(x) = x/ \ln |x|$, then

$$f'(0) = \lim_{h\to 0} \frac{h}{h \ln |h|} = \lim_{h\to 0} \frac{1}{\ln |h|} = 0,$$

so the derivative does exist. But

$$f(0+h) - f(0) - hf'(0) = \frac{h}{\ln |h|}$$

is larger than h^2, since the limit

$$\lim_{h\to 0} \frac{h}{h^2 \ln |h|} = \lim_{h\to 0} \frac{1}{h \ln |h|}$$

is infinite: the denominator tends to 0 as h tends to 0.

1.7.5 a. Compute the partial derivatives:

$$D_1 f \begin{pmatrix} x \\ y \end{pmatrix} = \frac{x}{\sqrt{x^2+y}} \quad \text{and} \quad D_2 f \begin{pmatrix} x \\ y \end{pmatrix} = \frac{1}{2\sqrt{x^2+y}}.$$

This gives

$$D_1 f \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{2}{\sqrt{2^2+1}} = \frac{2}{\sqrt{5}} \quad \text{and} \quad D_2 f \begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{1}{2\sqrt{2^2+1}} = \frac{1}{2\sqrt{5}}.$$

At the point $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$, we have $x^2 + y < 0$, so the function is not defined there, and neither are the partial derivatives.

b. Similarly, $D_1 f \begin{pmatrix} x \\ y \end{pmatrix} = 2xy$ and $D_2 f \begin{pmatrix} x \\ y \end{pmatrix} = x^2 + 4y^3$. This gives

$$D_1 f \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 4 \quad \text{and} \quad D_2 f \begin{pmatrix} 2 \\ 1 \end{pmatrix} = 4 + 4 = 8;$$

$$D_1 f \begin{pmatrix} 1 \\ -2 \end{pmatrix} = -4 \quad \text{and} \quad D_2 f \begin{pmatrix} 1 \\ -2 \end{pmatrix} = 1 + 4 \cdot (-8) = -31.$$

c. Compute

$$D_1 f\begin{pmatrix} x \\ y \end{pmatrix} = -y \sin xy$$
$$D_2 f\begin{pmatrix} x \\ y \end{pmatrix} = -x \sin xy + \cos y - y \sin y.$$

This gives

$$D_1 f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -\sin 2 \quad \text{and} \quad D_2 f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = -2\sin 2 + \cos 1 - \sin 1$$
$$D_1 f\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -2\sin 2 \quad \text{and} \quad D_2 f\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \sin 2 + \cos 2 - 2\sin 2 = \cos 2 - \sin 2$$

d. Since

$$D_1 f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{xy^2 + 2y^4}{2(x+y^2)^{3/2}} \quad \text{and} \quad D_2 f\begin{pmatrix} x \\ y \end{pmatrix} = \frac{2x^2y + xy^3}{(x+y^2)^{3/2}},$$

we have

$$D_1 f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{4}{2\sqrt{27}} \quad \text{and} \quad D_2 f\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \frac{10}{\sqrt{27}};$$
$$D_1 f\begin{pmatrix} 1 \\ -2 \end{pmatrix} = \frac{36}{10\sqrt{5}} \quad \text{and} \quad D_2 f\begin{pmatrix} 1 \\ -2 \end{pmatrix} = -\frac{12}{5\sqrt{5}}.$$

1.7.6 a. We have

$$\frac{\partial \vec{\mathbf{f}}}{\partial x}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} -\sin x \\ 2xy \\ 2x\cos(x^2-y) \end{bmatrix} \quad \text{and} \quad \frac{\partial \vec{\mathbf{f}}}{\partial y}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 \\ x^2+2y \\ -\cos(x^2-y) \end{bmatrix}.$$

b. Similarly,

$$\frac{\partial \vec{\mathbf{f}}}{\partial x}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} \\ y \\ 2y\sin xy \;\; \cos xy \end{bmatrix} \quad \text{and} \quad \frac{\partial \vec{\mathbf{f}}}{\partial y}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} \frac{y}{\sqrt{x^2+y^2}} \\ x \\ 2x\sin xy \;\; \cos xy \end{bmatrix}.$$

1.7.7 Just pile up the partial derivative vectors side by side:

a. $$\left[\mathbf{D}\vec{\mathbf{f}}\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{bmatrix} -\sin x & 0 \\ 2xy & x^2+2y \\ 2x\cos(x^2-y) & -\cos(x^2-y) \end{bmatrix}$$

b. $$\left[\mathbf{D}\vec{\mathbf{f}}\begin{pmatrix} x \\ y \end{pmatrix}\right] = \begin{bmatrix} \frac{x}{\sqrt{x^2+y^2}} & \frac{y}{\sqrt{x^2+y^2}} \\ y & x \\ 2y\sin xy \;\; \cos xy & 2x\sin xy \;\; \cos xy \end{bmatrix}.$$

1.7.8 a. $D_1 f_1 = 2x\cos(x^2+y)$, $D_2 f_1 = \cos(x^2+y)$, $D_2 f_2 = xe^{xy}$

b. 3×2.

1.7.9 a. The derivative is an $m \times n$ matrix

b. a 1×3 matrix (line matrix)

c. a 4×1 matrix (vector 4 high)