Math 8245 **Theory Homework 1** **PJW**

Date due: Wednesday September 19, 2018. In class on Friday September 21 we will grade your answers, so it is important to be present on that day, with your homework.

Some general questions

1. Let $N \lhd G = H \times K$. Prove that either N is abelian or N intersects one of the factors H or K nontrivially.

2. If $H \leq L \leq G$ and $N \lhd G$ show that the equations $HN = LN$ and $H \cap N = L \cap N$ imply that $H = L$.

3. a) (The modular law) Let H, K, and L be subgroups of G with $H \subseteq L$. Show that

$$HK \cap L = H(K \cap L).$$

   b) Suppose we remove the requirement in a) that $H \subseteq L$. Give an example to show that the conclusion need not hold.

4. Let G be a finite group with a normal subgroup H such that $(|H|, |G : H|) = 1$. Show that H is the unique subgroup of G having order $|H|$.
   [Hint: If K is another such subgroup, what happens to K in G/H?]

Semidirect and wreath products

5. Let G be a group, and consider the usual homomorphism $\theta : G \to \operatorname{Aut} G$ where $\theta(g)(x) = gxg^{-1}$, so $\theta(g)$ is conjugation by g. Using θ we may form the semidirect product $G \rtimes G$. Show that $G \rtimes G \cong G \times G$.
   [Hint: Look for a subgroup of $G \times G$ which acts on G via θ.]

6. Prove that the standard restricted wreath product $\mathbb{Z} \wr \mathbb{Z}$ is finitely generated but has a non-finitely generated subgroup. (By *standard* I mean the wreath product where the factor group acts on the base group by means of the regular permutation representation.)

7. Let

$$G = \{ \begin{pmatrix} 1 & a & b & c \\ 0 & 1 & d & e \\ 0 & 0 & 1 & f \\ 0 & 0 & 0 & 1 \end{pmatrix} \mid a, b, c, d, e, f \in \mathbb{Z}/2\mathbb{Z} \} \subseteq GL(4, 2).$$

   Show that $G \cong C_2 \wr (C_2 \times C_2)$ where the $C_2 \times C_2$ acts regularly on a set of size 4.
   [First show that $G = N \rtimes H$ where N is the subgroup specified by $a = f = 0$ and H is the subgroup specifed by $b = c = d = e = 0$.]

EXTRA QUESTIONS, not part of the homework assignment

8. Prove that the standard wreath product $C_2 \wr C_2$ is isomorphic to D_8.

9. Let S_G be the group of all permutations of G (the symmetric group on G), and observe that $\mathrm{Aut}(G)$ is a subgroup of S_G. Let $\lambda : G \to S_G$ be the homomorphism given by the left regular representation of G, so for each $g \in G$, $\lambda(g)$ is the permutation of G given by $\lambda(g)(x) = gx$, and let $\rho : G \to S_G$ be the homomorphism given by the right regular representation of G, so for each $g \in G$, $\rho(g)$ is the permutation of G given by $\rho(g)(x) = xg^{-1}$.
   (a) Show that $\langle \lambda(G), \mathrm{Aut}(G)\rangle = \langle \rho(G), \mathrm{Aut}(G)\rangle$ as subgroups of S_G, and they have the form $G \rtimes \mathrm{Aut}(G)$ (a group known as the *holomorph* of G).
   (b) Show that $N_{S_G}(\lambda(G)) = \langle \lambda(G), \mathrm{Aut}(G)\rangle$.
   (c) Deduce (for example) that

$$N_{S_8}(\langle(1,2)(3,4)(5,6)(7,8),(1,3)(2,4)(5,7)(6,8),(1,5)(2,6)(3,7)(4,8)\rangle) \cong (C_2 \times C_2 \times C_2) \rtimes GL(3,2).$$

[This question seems fairly hard, and you may wish to proceed using the following steps.
a) Establish the formula $\alpha\lambda(g)\alpha^{-1} = \lambda(\alpha(g))$ for all $\alpha \in \mathrm{Aut}\, G$ and $g \in G$.
b) Any $\beta \in N_{S_G}(\lambda(G))$ can be written $\beta = \lambda(g)\beta'$ for some $g \in G$, where $\beta'(1) = 1$.
c) Given $\gamma \in N_{S_G}(\lambda(G))$ there exists $\alpha \in \mathrm{Aut}(G)$ with $\gamma\lambda(g)\gamma^{-1} = \lambda(\alpha(g))$ for all $g \in G$. Deduce that $\alpha^{-1}\gamma \in C_{S_G}(\lambda(G))$.
d) Show that if $\delta \in C_{S_G}(\lambda(G))$ and $\delta(1) = 1$, then δ is the identity permutation of G.
e) Put the previous pieces together!]