Math 8246 **Homework 3** **PJW**
Date due: Friday March 29, 2019.

1. Let V be the Klein 4-group and let $G = \mathrm{Aut}(V) \cong S_3$ act on V in the natural fashion. Prove that $H^1(G, V) = 0$. [Show that in the semidirect product $E = V \rtimes G$, G is the normalizer of a Sylow 3-subgroup of E. Apply Sylow's Theorem to show all complements to V in E are conjugate.]

2. Let $1 \to M \xrightarrow{i} E \xrightarrow{p} G \to 1$ be a group extension in which M is abelian, and let $s : G \to E$ be a section for p, that is, a mapping which satisfies $ps = 1_G$. Thus each element of E can be written in the form $i(m) \cdot s(x)$ with m and x uniquely determined. The multiplication in E determines a function $f : G \times G \to M$ by

$$s(x) \cdot s(x') = if(x, x') \cdot s(xx'), \quad x, x' \in G.$$

   (i) Show that associativity of multiplication in E implies

$$xf(y, z) - f(xy, z) + f(x, yz) - f(x, y) = 0, \quad \text{for all } x, y, z \in G.$$

   A function f satisfying this condition is called a *factor set*.

   (ii) Show that factor sets form a group under $(f_1 + f_2)(x, x') = f_1(x, x') + f_2(x, x')$.

   (iii) Show that if $g : G \to M$ is any function then the function which sends (x, y) to $g(xy) - g(x) - xg(y)$ is a factor set.

   (iv) Show that if $s, s' : G \to E$ are two sections and f, f' the corresponding factor sets, then there is a function $g : G \to M$ with $f'(x, y) = f(x, y) + g(xy) - g(x) - xg(y)$.

   [In fact the quotient of the group of factor sets by the factor sets of the form g is isomorphic to $H^2(G, M)$ and we have gone some way towards showing from this point of view that this group bijects with equivalence classes of extensions.]

3. (a) Let G be a group with a presentation $G = \langle g_1, \ldots, g_d \mid a_1, \ldots, a_r \rangle$ and suppose that the abelianisation G/G' is the direct sum of a free abelian group of rank s and a finite group. Show that $H_2(G, \mathbb{Z})$ can be generated by no more than $r - d + s$ elements.
   (b) Show that the braid group on three strings $B_3 = \langle g_1, g_2 \mid g_1 g_2 g_1 = g_2 g_1 g_2 \rangle$ has trivial Schur multiplier. Show that $H_2(\mathbb{Z}^n, \mathbb{Z})$ can be generated by at most $\binom{n}{2}$ elements.

Questions on the Free Differential Calculus of R.H. Fox

Be sure to use things you know about derivations and augmentation ideals, rather than trying to derive results by bare hands from scratch. Derivations biject with homomorphisms from the augmentation ideal. The augmentation ideal is generated by elements $g_i - 1$ where the g_i generate G. The augmentation ideal of a free group is a free module of rank equal to the rank of the group.

4. Suppose that G is generated by elements $g_1, \ldots, g_n$, that is $G = \langle g_1, \ldots, g_n \rangle$, and let $d : G \to M$ be a derivation. Show that for each element $g \in G$ there exist elements $\lambda_i \in \mathbb{Z}G$ (which will depend on g) such that $d(g) = \sum \lambda_i d(g_i)$. Conclude that the $\mathbb{Z}$-linear span of $d(G)$ is the $\mathbb{Z}G$-submodule of M generated by $d(g_1), \ldots, d(g_n)$.

5. Apply the last question in the case of the derivation $d : G \to \mathbb{Z}G$ given by $d(g) = g - 1$, so that this defines $\lambda_1, \ldots, \lambda_n \in \mathbb{Z}G$. Show that now for any other derivation $e : G \to N$ we also have $e(g) = \sum \lambda_i e(g_i)$, this equation holding in N.

6. Let F be the free group on generators $g_1, \ldots, g_n$ and consider $d : F \to \mathbb{Z}F$ given by $d(g) = g - 1$. Show that the elements λ_i considered in the last questions are now uniquely determined. We will denote the element $\lambda_i \in \mathbb{Z}F$ by $\frac{\partial g}{\partial g_i}$. Thus

$$d(g) = \sum_i \frac{\partial g}{\partial g_i} d(g_i).$$

Show that
(i) the mapping $\frac{\partial}{\partial g_i} : F \to \mathbb{Z}F$ is a derivation,
(ii) $\frac{\partial g_j}{\partial g_i} = \delta_{ij}$.
The properties (i) and (ii) in fact characterize $\frac{\partial}{\partial g_i}$. Demonstrate this by computing $\frac{\partial}{\partial x}(yx^{-1}yx^2)$.
[The approach presented here is the 'free differential calculus' of R.H. Fox, and one of its uses can be found described in the book on knot theory by R.H. Crowell and R.H. Fox.]

7. Let $1 \to R \to F \to G \to 1$ be a presentation of G, so

$$G = \langle g_1 \ldots g_n \mid r_1, \ldots, r_m \rangle,$$

where R is the normal subgroup of F generated by $r_1, \ldots, r_m$. Show that

$$r_j - 1 = \sum_i \frac{\partial r_j}{\partial g_i}(g_i - 1), \quad j = 1, \ldots, m.$$

Deduce that in the start of the resolution

$$\begin{array}{ccccccccccc} \mathbb{Z}G^m & & \xrightarrow{\alpha} & & \mathbb{Z}G^n & & \longrightarrow & & \mathbb{Z}G & \xrightarrow{\epsilon} \mathbb{Z} \longrightarrow 0 \\ & \searrow & & \nearrow & & \searrow & & \nearrow & & \\ & & R/R' & & & & IG & & & \end{array}$$

where the basis vectors of $\mathbb{Z}G^m$ are mapped to the generators $r_j R'$ of R/R', the map α has matrix $\left(\frac{\partial r_j}{\partial g_i}\right)_{i,j}$.
Evaluate this matrix in the case of the group given by the presentation

$$G = \langle x, y \mid x^4 = 1 = y^4,\ x^2 = y^2,\ yxy^{-1} = x^3 \rangle.$$

[One might call this the *Jacobian matrix*. It appears in the definition of the Alexander polynomial of a knot.]

Extra questions: do not hand in!

8. Suppose that we have two commutative diagrams of group homomorphisms

$$\begin{array}{ccccccccc} 1 & \to & L & \xrightarrow{\gamma} & J & \to & G & \to & 1 \\ & & \big\downarrow \theta & & \big\downarrow \phi_i & & \big\downarrow 1_G & & \\ 1 & \to & M & \xrightarrow{\alpha_i} & E_i & \to & G & \to & 1 \end{array}$$

where $i = 1, 2$, the maps labeled without the suffix i are the same in both diagrams, L and M are abelian and the two rows are group extensions (i.e. short exact sequences of groups). Assume that the two module actions of G on M given by conjugation within E_1 and E_2 are the same. Show that the two bottom extensions are equivalent. [Hint: one way to proceed is to show that they are both equivalent to a third extension with middle group $E = (M \rtimes J)/\{(-\theta(x), \gamma(x)) \bigm| x \in L\}$. Do this by showing that there are homomorphisms $M \rtimes J \to E_i$, $i = 1, 2$, commuting with other specified homomorphism, and that these induce homomorphisms $E \to E_i$, which you then show to be isomorphisms.]

9. Let M be a normal subgroup of a group E, write $G = E/M$ and suppose that M is generated as a normal subgroup of E by elements $m_1, m_2, \ldots, m_s$.
   (i) Show that $m_1 M', \ldots, m_s M'$ generate M/M' as a $\mathbb{Z}G$-module.
   (ii) Show that $m_1 - 1, \ldots, m_s - 1$ generate $\mathbb{Z}E \cdot IM = \mathbb{Z}E \cdot IM \cdot \mathbb{Z}E$ as a 2-sided ideal of $\mathbb{Z}E$.

10. By considering the short exact sequence $0 \to \mathbb{Z} \to \mathbb{Z} \to \mathbb{Z}/|G|\mathbb{Z} \to 0$ show that if G is finite and $H_2(G, \mathbb{Z}) = 0$ then $H_2(G, \mathbb{Z}/|G|\mathbb{Z}) \cong G/G'$. (This suggests that it does not work to compute the Schur multiplier using a finite coefficient module.)