Math 8300 **Homework 3** **PJW**
Date due: Friday November 1, 2019

1. Let Q be the quiver $\begin{array}{ccccc} & & 2 & & \\ & \nearrow & & \searrow & \\ 1 & & \rightarrow & & 3 \end{array}$ of type $\tilde{A}_2$.

   (a) Show that $(2, 1, 1)$ is a real root of Q, and describe explicitly an indecomposable representation of Q over $K = \mathbb{Q}$ with this dimension vector (specifying the maps between the three different vector spaces by matrices).

   (b) Show that, over any field, Q has infinitely many isomorphism classes of indecomposable representations.

2. Describe all the abelian groups M that can appear in an essential epimorphism $L \to M$ of abelian groups in each of the following three cases:

   (a) $L = \mathbb{Z} \to M$,

   (b) $L = \mathbb{Z}/4\mathbb{Z} \to M$,

   (c) $L = \mathbb{Z}/6\mathbb{Z} \to M$.

3. Let A be a ring with a 1, and let V be an A-module. We write $\mathrm{End}_A(V)$ for the set of A-module homomorphisms $V \to V$. Two idempotents $e, f \in \mathrm{End}_A(V)$ are *orthogonal* if $ef = fe = 0$. The idempotent e is *primitive* if it cannot be written $e = e_1 + e_2$ where e_1, e_2 are (non-zero) orthogonal idempotents.

   (a) Show that an element $e \in \mathrm{End}_A(V)$ is idempotent if and only if e is the identity on restriction to the subspace $e(V)$ of V.

   (b) Show that direct sum decompositions $V = W_1 \oplus W_2$ as A-modules are in bijection with expressions $1 = e + f$ in $\mathrm{End}_A(V)$, where e and f are orthogonal idempotents.

   (c) Show that an idempotent $e \in \mathrm{End}_A(V)$ is primitive if and only if the submodule $e(V)$ of V is indecomposable as an A-module.

   (d) Suppose that V is semisimple with finitely many simple summands and let $e_1, e_2 \in \mathrm{End}_A(V)$ be idempotent elements. Show that $e_1(V) \cong e_2(V)$ as A-modules if and only if e_1 and e_2 are conjugate by an invertible element of $\mathrm{End}_A(V)$ (i.e. there exists an invertible A-module homomorphism $\alpha : V \to V$ such that $e_2 = \alpha e_1 \alpha^{-1}$).

4. A module U is said to be *uniserial* if it has a unique composition series.

(a) If U is the direct sum of two non-zero submodules, show that U is not uniserial.

(b) If U is indecomposable and has just two composition factors, show that U is uniserial.

(c) Give an example of a finite dimensional algebra A with a finite dimensional indecomposable module that is not uniserial. (Yes, we have had some in class.)

5. Show that the following conditions are equivalent for a module U that has a composition series.

   (a) U is uniserial (i.e. U has a unique composition series).

   (b) The set of all submodules of U is totally ordered by inclusion.

   (c) $\operatorname{Rad}^r U / \operatorname{Rad}^{r+1} U$ is simple for all r.

   (d) $\operatorname{Soc}^{r+1} U / \operatorname{Soc}^r U$ is simple for all r.

6. Let A be a finite dimensional algebra over a field and let U be an A-module. Write $\ell(U)$ for the Loewy length (radical length) of U.

   (a) Suppose V is a submodule of U. Show that $\ell(V) \leq \ell(U)$ and $\ell(U/V) \leq \ell(U)$. Show by example that we can have equality here even when $0 < V < U$.

   (b) Suppose that $U_1, \ldots, U_n$ are submodules of U for which $U = U_1 + \cdots + U_n$. Show that $\ell(U) = \max\{\ell(U_i) \mid 1 \leq i \leq n\}$.

7. Let A be a finite dimensional algebra. Notice that $\operatorname{Rad} A$, defined as the intersection of the maximal left ideals of A, is also the intersection of the maximal right ideals of A: both of these intersections are the unique nilpotent ideal with semisimple quotient.

   (a) If A is a semisimple algebra, show that A has the same number of simple left modules as simple right modules (up to isomorphism), and that their dimensions are the same.

   (b) Extend the last result to a general finite dimensional algebra A that is not semisimple: show that the number of simple left modules equals the number of simple right modules (up to isomorphism), and that their dimensions are the same.

8. Let A be a finite dimensional algebra over a field. Show that A is semisimple if and only if all finite dimensional A-modules are projective.