Math 8300 Representations of Finite Dimensional Algebras

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Chapter 1

Examples

1.1 Algebras

Let R be a commutative ring with a 1.

Definition 1.1.1. An (associative) R-*algebra* is a (not necessarily commutative) ring A with a 1, equipped with a (unital) ring homomorphism $R \to A$ whose image lies in the center of A.

In all our examples this homomorphism will be one-to-one, so that R is realized as a subring of the center of A. The polynomial ring $R[x_1, \ldots, x_n]$ and the matrix ring $M_n(R)$ are examples of R-algebras. The first is commutative, the second is not.

Definition 1.1.2. The opposite ring of a ring A is denoted A^{op}. This ring has the same elements of A and a new multiplication $a \cdot b := ba$. If A is an R-algebra, so is A^{op}.

As well as being a ring in its own right, an R-algebra A can be regarded as an R-module, where the action of R is given by multiplication within A. When R is a field, A is a vector space over R. To say that A is a finite dimensional algebra means that R is a field, and A is finite dimensional as a vector space over R.

Definition 1.1.3. A *homomorphism* of R-algebras $A \to B$ is a ring homomorphism that is also an R-module homomorphism. A *representation* of an R-algebra A is the same thing as an A-module.

This means that a representation of A is an abelian group M together with a ring homomorphism $A \to \text{End}_{\mathbb{Z}}(M)$, the ring of endomorphisms of M as an abelian group. Because of the ring homomorphism $R \to A$ we have, by composition, a ring homomorphism $R \to \text{End}_{\mathbb{Z}}(M)$, so that M also has the structure of an R-module.

Class Activity. Given an R-algebra A and an A-module M, then $\text{End}_R(M)$ is an R-algebra, and the homomorphism $A \to \text{End}_R(M)$ is a homomorphism of R-algebras. Is this obvious? Is it easy? Is there a difficulty? Would the answers to these questions change if $\text{End}_{\mathbb{Z}}(M)$ were asked about instead of $\text{End}_R(M)$? Is the definition of a

representation of A even correct? Should it require M to be an R-module, and $A \to$ $\text{End}_R(M)$ a homomorphism of R-algebras?

Why should we make the structure of a ring more complicated by introducing an extra ring R in the definition of an algebra? What mathematical or other reason do we have for considering algebras and their representations at all? What goals do we have in describing representations? So far, the definitions have been rather abstract. These points will be addressed, to some extent, by considering examples.

1.2 Representations of groups

This section is adapted from P.J. Webb, A course in finite group representation theory, Cambridge 2016.

Let G denote a finite group, and let R be a commutative ring with a 1. If V is an R-module we denote by $GL(V)$ the group of all invertible R-module homomorphisms $V \to V$. In case $V \cong R^n$ is a free module of rank n this group is isomorphic to the group of all non-singular $n \times n$ matrices over R, and we denote it by $GL(n, R)$ or $GL_n(R)$, or in case $R = \mathbb{F}_q$ is the finite field with q elements by $GL(n, q)$ or $GL_n(q)$. We point out also that unless otherwise stated, modules will be left modules and morphisms will be composed reading from right to left, so that matrices in $GL(n, R)$ are thought of as acting from the left on column vectors.

A *(linear) representation* of G (over R) is a group homomorphism

$$\rho : G \to GL(V).$$

In a situation where V is free as an R-module, on taking a basis for V we may write each element of $GL(V)$ as a matrix with entries in R and we obtain for each $g \in G$ a matrix $\rho(g)$. These matrices multiply together in the manner of the group and we have a *matrix representation* of G. In this situation the rank of the free R-module V is called the *degree* of the representation. Sometimes by abuse of terminology the module V is also called the representation, but it should more properly be called the *representation module* or *representation space* (if R is a field).

To illustrate some of the possibilities that may arise we consider some examples.

Example 1.2.1. For any group G and commutative ring R we can take $V = R$ and $\rho(g) = 1$ for all $g \in G$, where 1 denotes the identify map $R \to R$. This representation is called the *trivial representation*, and it is often denoted simply by its representation module R. Although this representation turns out to be extremely important in the theory, it does not at this point give much insight into the nature of a representation.

Example 1.2.2. A representation on a space $V = R$ of rank 1 is in general determined by specifying a homomorphism $G \to R^\times$. Here $R^\times$ is the group of units of R, and it is isomorphic to $GL(V)$. For example, if $G = \langle g \rangle$ is cyclic of order n and $k = \mathbb{C}$ is the field of complex numbers, there are n possible such homomorphisms, determined by $g \mapsto e^{\frac{2r\pi i}{n}}$ where $0 \le r \le n-1$. Another important example of a degree 1 representation

is the *sign representation* of the symmetric group S_n on n symbols, given by the group homomorphism which assigns to each permutation its sign, regarded as an element of the arbitrary ring R.

Example 1.2.3. Let $R = \mathbb{R}$, $V = \mathbb{R}^2$ and $G = S_3$. This group G is isomorphic to the group of symmetries of an equilateral triangle. The symmetries are the three reflections in the lines that bisect the equilateral triangle, together with three rotations.
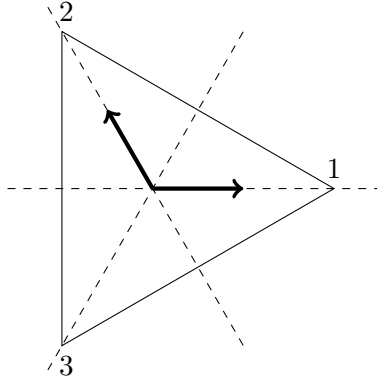
Positioning the center of the triangle at the origin of V and labeling the three vertices of the triangle as 1, 2 and 3 we get a representation

$$() \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$(1,2) \mapsto \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$$

$$(1,3) \mapsto \begin{bmatrix} -1 & 0 \\ -1 & 1 \end{bmatrix}$$

$$(2,3) \mapsto \begin{bmatrix} 1 & -1 \\ 0 & -1 \end{bmatrix}$$

$$(1,2,3) \mapsto \begin{bmatrix} 0 & -1 \\ 1 & -1 \end{bmatrix}$$

$$(1,3,2) \mapsto \begin{bmatrix} -1 & 1 \\ -1 & 0 \end{bmatrix}$$

where we have taken basis vectors in the directions of vertices 1 and 2, making an angle of $\frac{2\pi}{3}$ to each other. In fact these matrices define a representation of degree 2 over any ring R, because although the representation was initially constructed over $\mathbb{R}$ the matrices have integer entries, and these may be interpreted in every ring. No matter what the ring is, the matrices always multiply together to give a copy of S_3.

At this point we have constructed three representations of S_3: the trivial representation, the sign representation and one of dimension 2.

Example 1.2.4. Let $R = \mathbb{F}_p$, $V = R^2$ and let $G = C_p = \langle g \rangle$ be cyclic of order p generated by an element g. We see that the assignment

$$\rho(g^r) = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}$$

is a representation. In this case the fact that we have a representation is very much dependent on the choice of R as the field $\mathbb{F}_p$: in any other characteristic it would not work, because the matrix shown would no longer have order p.

We can think of representations in various ways. One of them is that a representation is the specification of an action of a group on an R-module, as we now explain. Given a representation $\rho : G \to GL(V)$, an element $v \in V$ and a group element $g \in G$ we get another module element $\rho(g)(v)$. Sometimes we write just $g \cdot v$ or gv for this element. This rule for multiplication satisfies

$$g \cdot (\lambda v + \mu w) = \lambda g \cdot v + \mu g \cdot w$$
$$(gh) \cdot v = g \cdot (h \cdot v)$$
$$1 \cdot v = v$$

for all $g \in G$, $v, w \in V$ and $\lambda, \mu \in R$. A rule for multiplication $G \times V \to V$ satisfying these conditions is called a *linear action* of G on V. To specify a linear action of G on V is the same thing as specifying a representation of G on V, since given a representation we obtain a linear action as indicated above, and evidently given a linear action we may recover the representation.

Another way to define a representation of a group is in terms of the group algebra. We define the *group algebra* RG (or $R[G]$) of G over R to be the free R-module with the elements of G as an R-basis, and with multiplication given on the basis elements by group multiplication. The elements of RG are the (formal) R-linear combinations of group elements, and the multiplication of the basis elements is extended to arbitrary elements using bilinearity of the operation. What this means is that a typical element of RG is an expression $\sum_{g \in G} a_g g$ where $a_g \in R$, and the multiplication of these elements is given symbolically by

$$\left(\sum_{g \in G} a_g g \right) \left(\sum_{h \in G} b_h h \right) = \sum_{k \in G} \left(\sum_{gh=k} a_g b_h \right) k.$$

More concretely, we exemplify this definition by listing some elements of the group algebra $\mathbb{Q}S_3$. We write elements of S_3 in cycle notation, such as $(1, 2)$. This group element gives rise to a basis element of the group algebra which we write either as $1 \cdot (1, 2)$, or simply as $(1, 2)$ again. The group identity element $()$ also serves as the identity element of $\mathbb{Q}S_3$. In general, elements of $\mathbb{Q}S_3$ may look like $(1, 2) - (2, 3)$ or $\frac{1}{5}(1, 2, 3) + 6(1, 2) - \frac{1}{7}(2, 3)$. Here is a computation:

$$(3(1, 2, 3) + (1, 2))(() - 2(2, 3)) = 3(1, 2, 3) + (1, 2) - 6(1, 2) - 2(1, 2, 3)$$
$$= (1, 2, 3) - 5(1, 2).$$

The group algebra RG is indeed an example of an R-algebra.

Example 1.2.5. If $G = \langle x \rangle$ is an infinite cyclic group then $\mathbb{Z}G = \mathbb{Z}[x, x^{-1}]$ is the ring of Laurent polynomials in x.

Example 1.2.6. If p is a prime number it follows from some algebraic number theory that the ring of integers $\mathbb{Z}[e^{2\pi i/p}] \cong \mathbb{Z}[X]/(1 + X + \cdots + x^{p-1})$ has rank $p - 1$ as a free abelian group. If $G = \langle x \rangle$ is cyclic of order p, there is a surjective ring homomorphism $\mathbb{Z}G \to \mathbb{Z}[e^{2\pi i/p}]$ specified by $x \mapsto e^{2\pi i/p}$. Its kernel is $\mathbb{Z}N$, where $N = \sum_{g \in G} g$. This relationship shows that $\mathbb{Z}[e^{2\pi i/p}]$-modules may be regarded as $\mathbb{Z}G$ modules, because $\mathbb{Z}G$ has an ideal that is free abelian of rank 1, with factor ring $\mathbb{Z}[e^{2\pi i/p}]$.

Having defined the group algebra, we may now instead define a representation of G over R to be a unital RG-module. The fact that this definition coincides with the previous ones is the content of the next proposition. We may refer to group representations as modules (for the group algebra).

Proposition 1.2.7. *A representation of G over R has the structure of a unital RG-module. Conversely, every unital RG-module provides a representation of G over R.*

Proof. Given a representation $\rho : G \to GL(V)$ we define a module action of RG on V by $(\sum a_g g)v = \sum a_g \rho(g)(v)$.

Given an RG-module V, the linear map $\rho(g) : v \mapsto gv$ is an automorphism of V and $\rho(g_1)\rho(g_2) = \rho(g_1 g_2)$ so $\rho : G \to GL(V)$ is a representation. $\qquad\square$

The group algebra gives another example of a representation, called the *regular representation*. In fact for any ring A we may regard A itself as a left A-module with the action of A on itself given by multiplication of the elements. We denote this left A-module by $_A A$ when we wish to emphasize the module structure, and this is the (left) regular representation of A. When $A = RG$ we may describe the action on $_{RG}RG$ by observing that each element $g \in G$ acts on $_{RG}RG$ by permuting the basis elements in the fashion $g \cdot h = gh$. Thus each g acts by a *permutation matrix*, namely a matrix in which in every row and column there is precisely one non-zero entry, and that non-zero entry is 1. The regular representation is an example of a *permutation representation*, namely one in which every group element acts by a permutation matrix.

Regarding representations of G as RG-modules has the advantage that many definitions we wish to make may be borrowed from module theory. Thus we may study RG-submodules of an RG-module V, and if we wish we may call them *subrepresentations* of the representation afforded by V. To specify an RG-submodule of V it is necessary to specify an R-submodule W of V that is closed under the action of RG. This is equivalent to requiring that $\rho(g)w \in W$ for all $g \in G$ and $w \in W$. We say that a submodule W satisfying this condition is *stable* under G, or that it is an *invariant submodule* or *invariant subspace* (if R happens to be a field). Such an invariant submodule W gives rise to a homomorphism $\rho_W : G \to GL(W)$ that is the subrepresentation afforded by W.

Example 1.2.8. 1. Let $C_2 = \{1, -1\}$ be cyclic of order 2 and consider the representation

$$\rho : C_2 \to GL(\mathbb{R}^2)$$

$$1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$-1 \mapsto \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

There are just four invariant subspaces, namely $\{0\}$, $\langle \binom{1}{0} \rangle$, $\langle \binom{0}{1} \rangle$, $\mathbb{R}^2$ and no others. The representation space $\mathbb{R}^2 = \langle \binom{1}{0} \rangle \oplus \langle \binom{0}{1} \rangle$ is the direct sum of two invariant subspaces.

Example 1.2.9. In Example 1.2.4 above we considered a representation ρ of $G = C_p = \langle g \rangle$ over $R = \mathbb{F}_p$, $V = R^2$ given by

$$\rho(g^r) = \begin{bmatrix} 1 & 0 \\ r & 1 \end{bmatrix}.$$

An elementary calculation shows that $\langle \binom{0}{1} \rangle$ is the only 1-dimensional invariant subspace, and so it is not possible to write the representation space V as the direct sum of two non-zero invariant subspaces (or submodules). This representation has only three invariant subspaces. In this example the group algebra $\mathbb{F}_p G \cong \mathbb{F}_p[X]/(X^p)$ with an isomorphism given by $g \leftrightarrow X + 1$. Every finite dimensional representation of this ring is a direct sum of modules $\mathbb{F}_p[X]/(X^r)$ where $1 \leq r \leq p$.

1.3   Representations of quivers

Definition 1.3.1. A *quiver* Q is a directed graph with vertices Q_0 and directed edges Q_1. For each directed edge $x \xrightarrow{\alpha} y$ the *origin* of α is $o(\alpha) = x$ and the *terminus* of α is $t(\alpha) = y$. A *representation* M of Q over R is the specification of

- an R module $M(x)$ for each vertex x in Q_0, and

- an R-module homomorphism $M(\alpha) : M(x) \to M(y)$ for each edge α in Q_1.

A *homomorphism* $f : M \to N$ of representations of Q is, for each vertex x, an R-module homomorphism $f_x : M(x) \to N(x)$ so that for every edge $x \xrightarrow{\alpha} y$ the diagram

$$\begin{array}{ccc} M(x) & \xrightarrow{M(\alpha)} & M(y) \\ f_x \downarrow & & \downarrow f_y \\ N(x) & \xrightarrow{N(\alpha)} & N(y) \end{array}$$

commutes. An *isomorphism* of representations is defined to be a homomorphism with a 2-sided inverse. For representations of quivers it is equivalent to require that each of the maps f_x be an isomorphism.

Example 1.3.2. For the quiver

$$Q_1 = \quad \bullet \circlearrowright \alpha$$

a representation over R is the specification of an R-module V and an R-linear map $\theta : V \to V$. A homomorphism from this representation to another representation $\phi : W \to W$ is an R-linear map $f : V \to W$ so that $\phi f = f\theta$. Two such representations are isomorphic if and only if there is an invertible such map f, so that $\phi = f\theta f^{-1}$. This requires $V \cong W$ as R-modules and, taking $V = W$ the condition is that ϕ and θ are conjugate by an invertible transformation. The classification of representations of Q_1 is thus the same problem as the classification of conjugacy classes of endomorphisms of an R-module which, when R is an algebraically closed field, is solved by Jordan canonical form (and over an arbitrary field by rational canonical form).

Example 1.3.3. For the quiver

$$Q_2 = \quad y\bullet \xleftarrow{\ \beta\ } \bullet x$$

a representation is a diagram of R-modules $M(y) \xleftarrow{\beta_*} M(x)$. Three such diagrams are $S_y = R \xleftarrow{0} 0$, $S_x = 0 \xleftarrow{0} R$ and $V = R \xleftarrow{1} R$. There are homomorphisms of representations $0 \to S_y \to V \to S_x \to 0$, where 0 denotes the representation $0 \xleftarrow{0} 0$.

Associated to a finite quiver Q there is an R-algebra, called the path algebra, which we now define.

Definition 1.3.4. A *path* in the quiver Q is a list of edges $p = (\alpha_n, \alpha_{n-1}, \ldots, \alpha_1)$ with $t(\alpha_i) = o(\alpha_{i+1})$ for each i with $1 \le i \le n-1$. Such a path has length n and passes through $n+1$ vertices (possibly with repeats). By convention we include a path of length 0 from x to x, for each vertex x of Q. This is an empty list of edges; we write 1_x for the empty list at x.

The *path algebra* of Q is the free R module with basis the paths in Q. We define a multiplication on Q by defining it on the basis elements and extending by R-linearity. For basis elements p_1 and p_2, if the end point of p_1 is the same as the starting point of p_2 we define the product of these elements to be the path $p_2 p_1$ obtained by concatenating the paths; and otherwise the product is zero.

Example 1.3.5. For the quivers

$$Q_1 = \quad \bullet \circlearrowright \alpha \qquad\qquad Q_2 = \quad y\bullet \xleftarrow{\ \beta\ } \bullet x$$

the path algebra of Q_1 is the polynomial ring $R[\alpha]$, whereas the path algebra of Q_2 is a free R-module of rank 3 with basis the three paths $1_x, 1_y, \beta$. The multiplications of these elements are $1_x 1_x = 1_x$, $1_y 1_y = 1_y$, $1_y \beta = \beta$, $\beta 1_x = \beta$ and all other products between the basis elements are zero. We see that this algebra is isomorphic to the subalgebra

of $M_2(R)$ consisting of the lower triangular matrices. This is because this algebra has a basis the matrices E_{11}, E_{21}, E_{22} with products $E_{11}E_{11} = E_{11}$, $E_{22}E_{22} = E_{22}$, $E_{22}E_{21} = E_{21}$, $E_{21}E_{11} = E_{21}$ and all other products between these matrices are zero.

Proposition 1.3.6. *Let Q be a finite quiver. The path algebra of Q is an associative R-algebra.*

Proof. The issues are that multiplication is associative, and the algebra has a multiplicative identity. Associativity follows because paths are lists of edges, so they are equal if and only if they have the same terms in the list. For paths that can be concatenated this means that it makes no difference if we multiply three paths by concatenating the first two and then the third, or by concatenating the last two and then the first. In other situations associativity follows because the product is zero. The identity element of the path algebra is $\sum_{x \in Q_0} 1_x$. $\qquad\square$

Proposition 1.3.7. *Representations of a finite quiver may be identified with unital modules for the path algebra, and under this identification homomorphisms of quiver representations are module homomorphisms.*

Proof. The correspondence is that for each representation M of Q we construct the module for the path algebra $\bigoplus_{x \in Q_0} M(x)$. Conversely, for each module L for the path algebra we associate the representation of the quiver with $M(x) := 1_x L$. For each edge $\alpha \in Q_1$ we associate the R-module homomorphism $M(\alpha) : M(x) \to M(y)$ that is multiplication by α. $\qquad\square$

This means that we can transport concepts from module theory to the theory of representations of quivers, and speak of subrepresentations, quotient representations, and so on.

Example 1.3.8. In Example 1.3.3, when R is a field the representations S_y and S_x are simple. The sequence of representations in that example is a short exact sequence. The representation V has precisely three subrepresentations.

1.4 Representations of posets

Definition 1.4.1. A *partially ordered set* or *poset* is a set $\mathcal{P}$ with a relation $x \leq y$ satisfying

- $x \leq x$,

- $x \leq y$ and $y \leq z$ implies $x \leq z$, and

- $x \leq y$ and $y \leq x$ implies $x = y$.

A *representation* M of a poset $\mathcal{P}$ is the specification of an R-module $M(x)$ for each element $x \in \mathcal{P}$, and for each pair of comparable elements $x \leq y$ there is an R-linear

map $M(x) \to M(y)$. These R-linear maps should have the property that if $x \leq y \leq z$ then the composite of the maps $M(x) \to M(y) \to M(z)$ must equal the specified map $M(x) \to M(z)$.

Example 1.4.2. If $\mathcal{P}$ is the poset with elements x, y and $x < y$, its representations are the same as the representations of the quiver Q_2 in Example 1.3.5.

At this point we may be expecting the definition of an algebra whose modules are the same thing as representations of the poset. This algebra is the *incidence algebra* of the poset, or else its opposite, depending on how the incidence algebra is defined. Rather than make these definitions we move to the more general situation of categories, which subsumes the constructions given so far.

1.5 Representations of categories

In this section we present the three kinds of examples we have seen so far as instances of a single construction. The algebras that arise provide a good range of examples of natural interest. Furthermore, the language of category theory will be extremely useful to us.

Definition 1.5.1. A *category* $\mathcal{C}$ consists of three ingredients: a class $\text{Ob}(\mathcal{C})$ of *objects*, a set of *morphisms* $\text{Hom}_{\mathcal{C}}(x, y)$ for each pair of objects x and y, and a composition mapping $\text{Hom}_{\mathcal{C}}(y, z) \times \text{Hom}_{\mathcal{C}}(x, y) \to \text{Hom}_{\mathcal{C}}(x, z)$ denoted $(f, g) \mapsto fg$. These ingredients are subject to the following axioms:

1. Hom sets are pairwise disjoint; that is, each $f \in \text{Hom}_{\mathcal{C}}(x, y)$ has a unique *domain* x and a unique *codomain* or *target* y;

2. for each object x there is an identity morphism $1_x \in \text{Hom}_{\mathcal{C}}(x, x)$ such that $f1_x = f$ and $1_y f = f$ for all $f : x \to y$;

3. composition is associative.

This definition captures many familiar examples: the category of sets, the category of groups, the category of topological spaces, and so on. In each case the morphisms in the category must be the morphisms appropriate for the structure we are considering: continuous maps for topological spaces, group homomorphisms for groups, all mappings of sets for the category of sets.

Definition 1.5.2. We will be particular concerned with A-Mod, the category of all left A-modules, and A-mod, the category of all finitely generated left A-modules, with module homomorphisms as the morphisms.

In these examples the collection of objects does not form a set. When the objects of $\mathcal{C}$ do form a set we call $\mathcal{C}$ a *small category*. Apart from these examples, many of our categories of interest will be finite, and the notion of a category can be regarded as a combinatorial construction.

Definition 1.5.3. A morphism $f : x \to y$ is said to be an *isomorphism*, or be *invertible*, if there is a morphism $g : y \to x$ so that $fg = 1_y$ and $gf = 1_x$. In that case the objects x and y are said to be *isomorphic*.

Example 1.5.4. A *monoid* is a set with an associative binary operation and an identity element. Every group and, more generally, every monoid can be regarded as a category with a single object, in which the elements of the monoid are the morphisms, and composition of morphisms is given by multiplication in the monoid. The monoid is realized as the *endomorphism monoid* of the object. In the other direction, for any object x in a category $\mathcal{C}$, the set of endomorphisms $\text{End}_{\mathcal{C}}(x)$ is a monoid. Thus monoids can be regarded as categories with a single object, and groups can be regarded as categories with a single object in which all the morphisms are invertible.

Example 1.5.5. Every poset $\mathcal{P}$ may be regarded as a category where the elements of $\mathcal{P}$ are the objects, and there is a single morphism $x \to y$ if and only if $x \le y$. Each category where there is at most one morphism between each pair of objects may be regarded as a preordered set, where for objects x and y we define $x \le y$ if there is a morphism $x \to y$. A *preordered set* is a set with a transitive reflexive binary relation $\le$ (we do not require the further condition that $x \le y$ and $y \le x$ imply $x = y$ that makes a preordered set a poset).

Example 1.5.6. Given a quiver Q we may form the *free category* $F(Q)$ on Q. The objects of $F(Q)$ are the vertices of Q, and for objects x and y the set of homomorphisms $\text{Hom}_{F(Q)}(x, y)$ is the set of all paths from x to y. Composition of morphisms is concatenation of paths. This construction may be found in MacLane's book [5]. Thus the quiver Q_1 in Example 1.3.5 determines a free category $F(Q_1)$ with infinitely many morphisms, and the free category $F(Q_2)$ has two objects and three morphisms.

Just as we introduced the group algebra of a group and the path algebra of a quiver, we now introduce the category algebra of a category. This generalizes the previous constructions.

Definition 1.5.7. The *category algebra* $R\mathcal{C}$ of a category $\mathcal{C}$ over R is defined to be the free R-module with the morphisms of the category $\mathcal{C}$ as a basis. The product of morphisms α and β as elements of $R\mathcal{C}$ is defined to be

$$\alpha\beta = \begin{cases} \alpha \circ \beta & \text{if } o(\alpha) = t(\beta), \\ 0 & \text{otherwise,} \end{cases}$$

and this product is extended to the whole of $R\mathcal{C}$ using bilinearity of multiplication.

Example 1.5.8. If $\mathcal{C}$ is a group, that is a category with one object in which every morphism is invertible, the category algebra $R\mathcal{C}$ is the group algebra.

Example 1.5.9. The category algebra of a partially ordered set (or rather, its opposite) is the *incidence algebra* of the poset. This may be taken as a definition of the incidence algebra.

Example 1.5.10. Starting with a quiver Q we may form the free category $F(Q)$. Now the category algebra RFQ is the same as the path algebra of Q.

Our next step is to introduce the notion of a representation of a category, generalizing the notions of representations of groups, posets, and quivers. The concept is expressed in terms of a functor.

Definition 1.5.11. If $\mathcal{C}$ and $\mathcal{D}$ are categories we define a *functor* $T : \mathcal{C} \to \mathcal{D}$ to be a function such that

1. if $x \in \mathrm{Ob}(\mathcal{C})$ then $T(x) \in \mathrm{Ob}(\mathcal{D})$,

2. if $\alpha : x \to y$ in $\mathcal{C}$, then $T(\alpha) : T(x) \to T(y)$ in $\mathcal{D}$,

3. if $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ in $\mathcal{C}$, then $T(x) \xrightarrow{T(\alpha)} T(y) \xrightarrow{T(\beta)} T(z)$ in $\mathcal{D}$ and $T(\beta\alpha) = T(\beta)T(\alpha)$,

4. $T(1_x) = 1_{T(x)}$ for every $x \in \mathrm{Ob}(\mathcal{C})$.

Many familiar operations on the objects of categories are examples of functors. Here are some.

Example 1.5.12. If x is an object of $\mathcal{C}$ we get a functor $\mathrm{Hom}_{\mathcal{C}}(x, -) : \mathcal{C} \to \mathrm{Set}$ that sends each object y to the set $\mathrm{Hom}_{\mathcal{C}}(x, y)$ and each morphism $\beta : y \to z$ to the mapping of sets $\mathrm{Hom}_{\mathcal{C}}(x, \beta) = \beta_* : \mathrm{Hom}_{\mathcal{C}}(x, y) \to \mathrm{Hom}_{\mathcal{C}}(x, z)$ that sends $\alpha : x \to y$ to $\beta\alpha : x \to z$.

Example 1.5.13. If X is a set we may form the free group $F(X)$ on X. For each mapping of sets $\alpha : X \to Y$ there is a unique group homomorphism $\alpha_* : F(X) \to F(Y)$ extending α. The construction of a free group is a functor $\mathrm{Set} \to \mathrm{Group}$.

Sometimes we may have the specification of a functor $T : \mathcal{C} \to \mathcal{D}$ except that the order of composition of morphisms is reversed by T, so that $T\beta\alpha) = T(\alpha)T(\beta)$. Such is the case with the functor $\mathrm{Hom}_{\mathcal{C}}(-, x) : \mathcal{C} \to \mathrm{Set}$. We can describe such a functor using the opposite of a category.

Definition 1.5.14. For each category $\mathcal{C}$, the *opposite category* $\mathcal{C}^{\mathrm{op}}$ has the same objects as $\mathcal{C}$ and for each morphism $\alpha : x \to y$ in $\mathcal{C}$ there is a formally defined morphism $\hat{\alpha} : y \to x$ in $\mathcal{C}^{\mathrm{op}}$. The rule of composition in $\mathcal{C}^{\mathrm{op}}$ is that if $x \xrightarrow{\alpha} y \xrightarrow{\beta} z$ in $\mathcal{C}$ then $\hat{\alpha}\hat{\beta} := \widehat{\beta\alpha} : z \to x$. We say that a functor $\mathcal{C}^{\mathrm{op}} \to \mathcal{D}$ is a *contravariant* functor on $\mathcal{C}$, and that the usual functors $\mathcal{C} \to \mathcal{D}$ are *covariant*.

Example 1.5.15. $\mathrm{Hom}_{\mathcal{C}}(-, x)$ is a contravariant functor $\mathcal{C} \to \mathrm{Set}$.

Class Activity. Think of some other functors. Can you think of

- a functor Finite Categories $\to$ Algebras?

- a functor Quivers $\to$ Categories?

These questions presuppose that there is even a category called Finite Categories, and a category called Quivers. Have you been told what they are? What information is missing? What do you need to check to verify that you really do have a functor of the kind asked about?

Definition 1.5.16. When $\mathcal{C}$ is a small category and R is a commutative ring with a 1 we define a *representation* of $\mathcal{C}$ over R to be a functor $F : \mathcal{C} \to R$-Mod where R-Mod is the category of R-modules.

Example 1.5.17. When $\mathcal{C}$ is a group or a poset, this definition coincides with the definition given before of a representation of these structures. When Q is a quiver, a representation of Q is the same thing as a representation of the free category $F(Q)$. In the cases of groups and quivers we have seen that representations may be regarded as modules for the group algebra, and for the path algebra.

We now show that for categories in general, modules for the category algebra may be regarded as representations of the category. This was observed by Mitchell, and we give only a special case of his result for categories with finitely many objects.

Proposition 1.5.18. *[Mitchell [6]] Let $\mathcal{C}$ be a category with finitely many objects. The categories of representations of $\mathcal{C}$ over R and of $R\mathcal{C}$-modules are equivalent.*

In stating this result we have used two concepts that have not yet been introduced. The first is that the homomorphisms of representations of $\mathcal{C}$ have not been defined (they are natural transformations of functors) and the second concept is that of an equivalence of categories. This could be avoided by taking an informal approach that representations of $\mathcal{C}$ over R are 'the same thing as' modules for $R\mathcal{C}$, which is what is often done with group algebras and path algebras. The idea of the proof is the same as what we have seen in these cases.

Proof. Given a representation $M : \mathcal{C} \to R$-mod we obtain an $R\mathcal{C}$-module $r(M) = \bigoplus_{x \in \text{Ob}\mathcal{C}} M(x)$ where the action of a morphism $\alpha : y \to z$ on an element $u \in M(x)$ is to send it to $M(\alpha)(u)$ if $x = y$ and zero otherwise. Conversely, given an $R\mathcal{C}$-module U, we define a functor $M = s(U) : \mathcal{C} \to R$-mod by specifying M at an object x in $\mathcal{C}$ as $M(x) = 1_x U$, where 1_x is the identity morphism at x. If $\alpha : x \to z$ is a morphism in $\mathcal{C}$ and $u \in 1_x U$ we define $M(\alpha)(u) = \alpha u$. The two operations r and s are functors, and they are inverse equivalences of categories. $\square$

Homomorphisms between representations of a category $\mathcal{C}$ could be defined to be $R\mathcal{C}$-module homomorphisms, but we can also define them directly, by analogy with the definition for representations of a quiver. If M and N are representations of $\mathcal{C}$ over R a homomorphism $\theta : M \to N$ is the specification for each object x of $\mathcal{C}$ of an R-module homomorphism $\theta_x : M(x) \to N(x)$ so that: for every morphism $\alpha : x \to y$ in $\mathcal{C}$ the following diagram commutes:

$$
\begin{array}{ccc}
M(x) & \xrightarrow{M(\alpha)} & M(y) \\
\theta_x \downarrow & & \downarrow \theta_y \\
N(x) & \xrightarrow{N(\alpha)} & N(y)
\end{array}
$$

Regarding representations of a quiver as representations of the free category of the quiver, this definition coincides with the previously given definition for quivers. Such mappings θ are a list of homomorphisms $(\theta_x)_{x \in \mathrm{Ob}\mathcal{C}}$ and they have a special name.

Definition 1.5.19. Let $M, N : \mathcal{C} \to \mathcal{D}$ be functors. A *natural transformation* $\theta : M \to N$ is the specification, for each $x \in \mathrm{Ob}\mathcal{C}$, of a morphism $\theta_x : M(x) \to N(x)$ in $\mathcal{D}$ so that for every morphism $\alpha : x \to y$ in $\mathcal{C}$ the following diagram commutes:

$$
\begin{array}{ccc}
M(x) & \overset{M(\alpha)}{\longrightarrow} & M(y) \\
\theta_x \downarrow & & \downarrow \theta_y \\
N(x) & \overset{N(\alpha)}{\longrightarrow} & N(y)
\end{array}
$$

Example 1.5.20. Regarding representations of $\mathcal{C}$ as functors $M : \mathcal{C} \to R$-Mod, the homomorphisms $M \to N$ are the natural transformations between these functors.

Example 1.5.21. Let $\mathrm{Fun}(\mathcal{C}, \mathcal{D})$ denote the set of functors from $\mathcal{C}$ to $\mathcal{D}$ where $\mathcal{C}, \mathcal{D}$ are small categories. This set is the set of homomorphisms in the category of small categories. The set has further structure, in that it is also a category: the objects in this category are the functors from $\mathcal{C}$ to $\mathcal{D}$ and the morphisms are the natural transformations. Composition of natural transformations is defined pointwise on objects, so that if $\theta : F \to G$ and $\psi : G \to H$ then $\psi\theta : F \to H$ is the natural transformation with $(\psi\theta)_x = \psi_x\theta_x : F(x) \to H(x)$ for each $x \in \mathrm{Ob}\mathcal{C}$. There is an identity natural transformation $1_F : F \to F$, defined to be $(1_F)_x := 1_{F(x)} : F(x) \to F(x)$ at x. Thus the category of small categories has three kinds of pieces of information: objects (the categories), morphisms (the functors) and natural transformations (morphisms between the functors). It is an example of a *2-category*, which we do not define here. In this terminology, categories as we have defined them are 1-categories.

Definition 1.5.22. Two functors $M, N : \mathcal{C} \to \mathcal{D}$ are *naturally isomorphic,* written $M \simeq N$, if they are isomorphic in the functor category, which is to say there are natural transformations $\theta : M \to N$ and $\psi : N \to M$ so that $\psi\theta = 1_M$ and $\theta\psi = 1_N$. Two categories $\mathcal{C}, \mathcal{D}$ are said to be *isomorphic* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ so that $GF = 1_{\mathcal{C}}$ and $FG = 1_{\mathcal{D}}$, the identity functors on $\mathcal{C}$ and $\mathcal{D}$. This is a strong condition, and often functors we construct in practice are not strictly inverse to each other, but only up to natural isomorphism. We say that categories $\mathcal{C}, \mathcal{D}$ are *(naturally) equivalent* if there are functors $F : \mathcal{C} \to \mathcal{D}$ and $G : \mathcal{D} \to \mathcal{C}$ so that $GF \simeq 1_{\mathcal{C}}$ and $FG \simeq 1_{\mathcal{D}}$.

Example 1.5.23. This notion of equivalence of categories is the one used in Mitchell's theorem 1.5.18. The two functors r and s that appear in the proof of that theorem do not compose to give the identity, but their composites are naturally equivalent to the identity.

1.6   Representations of algebras in general

The point about all this is to show that by studying representations of finite dimensional algebras we are studying many different examples of natural interest. By regarding these examples as representations of an algebra we can use the language of module theory. The category of representations of a category $\mathcal{C}$ over R is *abelian*, which we will not define here, but it implies that we may form subrepresentations, and quotient representations of a representation by a subrepresentation, namely, submodules and factor modules for the category algebra. We have the notion of isomorphism of representations, as well as simple, projective and injective representations, and direct sums of representations.

For representations of the special cases groups, posets and quivers, these concepts from module theory might be less obvious. We might say that 'u is an element of a representation M of a category $\mathcal{C}$' and this will mean that u is an element of the module that corresponds to M, namely $\bigoplus_{x \in \mathrm{Ob}\mathcal{C}} M(x)$. We may speak of the subfunctor generated by a set of elements of M, and this means the submodule generated by those elements, or equally the intersection of all the subfunctors that contain the elements. Thus we may say that a functor is 'generated by its value at an object x', for example, to mean that it is the smallest subfunctor whose value at x is the given functor, or that the subfunctor is finitely generated. Such concepts are translated from modules for the category algebra to representations of the category. In some books on group representation theory, for example, the algebraic condition that defines a homomorphism of representations is written out explicitly, and two representations that are isomorphic are also said to be equivalent (a term we will not use in this context). We do not need to do this because we can transport these concepts from module theory.

We have given examples of algebras that are category algebras because such examples are of wide interest, and it is valuable to see that representations of different kinds of algebraic structure are all instances of the same construction. Not all finite dimensional algebras are category algebras. Category algebras have a *multiplicative basis*, namely a basis for which the product of any two elements is either a member of the basis or zero. Many algebras do not have a multiplicative basis.

Exercise 1.6.1. Show that $\mathbb{Q}[X]/(x^2)$ is not the category algebra of a category. This can be done by considering how many morphisms a category must have for this to be the category algebra, listing the categories with that number of morphisms, and showing that none of these categories have the correct category algebra.

Class Activity. Three questions were asked near the start of this chapter. To what extent have they been answered? The questions were:

- Why should we make the structure of a ring more complicated by introducing an extra ring R in the definition of an algebra?

- What mathematical or other reason do we have for considering algebras and their representations at all?

- What goals do we have in describing representations?

Chapter 2

Gabriel's theorem

We follow Chapter 1 of the book
[DDPW] B.Deng, J. Du, B. Parshall and J. Wang, *Finite dimensional algebras and quantum groups*, AMS 2008.

Concordance of notation: in Definition 1.1 the tail and head of an arrow ρ are $t\rho$ and $h\rho$, whereas we write $o\rho$ and $t\rho$ for the origin and terminus of ρ. In Definition 1.3 a representation of a quiver Q is written $V = (V_i, V_\rho)$, whereas we write $V(i)$ for the R-module at vertex i and $V(\rho) : V(i) \to V(j)$ for the R-module homomorphism associated to the arrow $i \xrightarrow{\rho} j$.

We study the start of section 1.2 up to part (1) of Lemma 1.8; the statement of Theorem 1.11; most of section 1.3 up to Examples 1.12 (2); then we start section 1.4.

At the start of section 1.3, we observe by a geometrical argument that if $v \in \mathbb{R}^n$ is a vector with $(v, v) = 2$, where $(-, -)$ is the usual bilinear form, and r_v denotes reflection in the hyperplane perpendicular to v, then $r_v(x) = x - (x, v)v$. This motivates the definition of the reflection r_i as $r_i(x) = x - (x, \alpha_i)\alpha_i$.

Proposition 2.0.1. *Suppose that Q is a quiver without loops.*

1. *The formula $r_i(\alpha_j) = \alpha_j - c_{i,j}\alpha_i$ can also be written*

$$r_i(x)_j = \begin{cases} x_j & \text{if } j \neq i \\ -x_i + \sum_{k, \text{edges between } i,k} x_k & \text{if } i = j \end{cases}$$

2. *$r_i^2 = 1$,*

3. *$(r_i(x), r_i(y)) = (x, y)$ always.*

Proof. (1) The proof of this formula is that

$$r_i(x) = x - \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} C_Q \begin{bmatrix} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{bmatrix}$$

$$= x - \begin{bmatrix} x_1 & \cdots & x_n \end{bmatrix} \begin{bmatrix} c_{1,i} \\ \vdots \\ c_{n,i} \end{bmatrix}$$

$$= \begin{bmatrix} x_1 \\ \vdots \\ x_i - (x_1 c_{1,i} + \cdots + x_n c_{n,i}) \\ \vdots \\ x_n \end{bmatrix}$$

and the expression in row i agrees with the desired formula because $c_{i,i} = 2$.

(2) This comes from direct calculation:

$$\begin{aligned}
r_i(r_i(x)) &= r_i(x) - (r_i(x), \alpha_i,)\alpha_i \\
&= x - (x, \alpha_i)\alpha_i - ((x - (x, \alpha_i)\alpha_i), \alpha_i)\alpha_i \\
&= x + (-(x, \alpha_i) - (x, \alpha_i) + (x, \alpha_i)(\alpha_i, \alpha_i))\alpha_i \\
&= x.
\end{aligned}$$

The proof of (3) is similarly done by calculation. $\qquad\square$

We define the *Weyl group* $W(Q)$, the *real roots* and the *positive roots*. Beyond this we do not need to define the root system.

Example 2.0.2. This is a special case of Example 1.12(2). We let $Q = \alpha_1 \bullet \to \bullet \alpha_2$, so that $C_Q = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$. This defines a positive definite inner product, and the angle between α_1 and α_2 is

$$\frac{(\alpha_1, \alpha_2)}{\sqrt{(\alpha_1, \alpha_1)(\alpha_2, \alpha_2)}} = \frac{-1}{2}.$$

We draw α_1 and α_2 at an angle of $120°$ and find that the real roots are the vertices of a regular hexagon and that $W(Q) \cong S_3$.

We now jump to section 1.4. Here are some preliminary lemmas for Theorem 1.18 of [DDPW]. The first is about kernels and cokernels. We could take the cokernel of a module homomorphism $A \xrightarrow{\alpha} B$ to be a homomorphism isomorphic to the quotient map $B \to B/\alpha(A)$. The first part of the next lemma says that this construction implies the categorical property that can also be taken to define the cokernel.

Lemma 2.0.3. *Let*

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C$$
$$\downarrow{\gamma}$$
$$D$$

be a diagram of modules for a ring. Suppose that $\beta = \operatorname{Coker}\alpha$ and $\gamma\alpha = 0$.

1. *There exists a unique homomorphism $\epsilon : C \to D$ so that $\gamma = \epsilon\beta$.*

2. *If in addition $A \xrightarrow{\alpha} B = \operatorname{Ker}\gamma$ then ϵ is one-to-one.*

There is also a dual version of this lemma to do with kernels.

Definition 2.0.4. We define what it means for a vertex k of Q to be a *sink* or a *source*. We also define the quiver $s_k Q$, obtained from Q by reversing all the arrows incident with k. Evidently $s_k s_k Q = Q$. If k is a sink of Q and V is a representation of Q there is a map of vector spaces

$$\xi_k : \bigoplus_{\alpha, t(\alpha) = k} V(o(\alpha)) \to V(k)$$

where the component maps are the $V(\alpha)$. Similarly, if k is a source there is a map

$$\gamma_k : V(k) \to \bigoplus_{\alpha, o(\alpha) = k} V(t(\alpha))$$

with component maps $V(\alpha)$.

Definition 2.0.5. We now define the *reflection functors* $\mathcal{R}_k^+, \mathcal{R}_k^-$: KFQ-mod $\to$ KFs_kQ-mod. The first of these, $\mathcal{R}_k^+$, is defined when k is a sink and the second, $\mathcal{R}_k^-$, is defined when k is a source. The definitions are these: for a sink k,

$$\mathcal{R}_k^+(V)(i) = \begin{cases} V(i) & \text{if } i \neq k, \\ \operatorname{Ker}\xi_k & \text{if } i = k. \end{cases}$$

When k is a source,

$$\mathcal{R}_k^-(V)(i) = \begin{cases} V(i) & \text{if } i \neq k, \\ \operatorname{Coker}\gamma_k & \text{if } i = k. \end{cases}$$

This is the definition on objects of KFQ-mod and, as explained in [DDPW], they are also defined on morphisms, making them functors. These functors are K-linear, meaning that they preserve linear combinations of morphisms, and they send direct sums of modules to direct sums.

Example 2.0.6. When $Q = \alpha_1 \bullet \to \bullet \alpha_2$ we compute $\mathcal{R}_2^+$, and compose it with $\mathcal{R}_2^-$ for $s_2 Q$.

The next result is a replacement for Theorem 1.18.

Theorem 2.0.7. *Let V be an indecomposable representation of the quiver Q over a field K.*

1. *If k is a sink then either the map $\xi_k : \bigoplus_{\alpha,t(\alpha)=k} V(o(\alpha)) \to V(k)$ is surjective or $V \cong S_k$ is simple. Furthermore $\mathcal{R}_k^+(V) = 0$ if and only if $V \cong S_k$. When $V \not\cong S_k$ there is a natural isomorphism $\mathcal{R}_k^- \mathcal{R}_k^+(V) \cong V$ and $\boldsymbol{dim}\mathcal{R}_k^+(V) = r_k \boldsymbol{dim}(V)$.*

2. *If k is a source then either the map $\gamma_k : V(k) \to \bigoplus_{\alpha,o(\alpha)=k} V(t(\alpha))$ is injective or $V \cong S_k$ is simple. Furthermore $\mathcal{R}_k^-(V) = 0$ if and only if $V \cong S_k$. When $V \not\cong S_k$ there is a natural isomorphism $\mathcal{R}_k^+ \mathcal{R}_k^-(V) \cong V$ and $\boldsymbol{dim}\mathcal{R}_k^-(V) = r_k \boldsymbol{dim}(V)$.*

Proof. The idea is the same as in the proof of Theorem 1.18.

(1) Choose a vector space decomposition $V(k) = W_1 \oplus W_2$ where $W_1 = \operatorname{Im} \xi_k$ and W_2 is some vector space complement to W_1. We get a decomposition $V = V_1 \oplus V_2$ where $V_i(j) = V(j)$ if $j \neq k$, and $V_i(k) = W_i$, for $i = 1, 2$. If V is indecomposable then either $V_2 = 0$, in which case ξ_k is surjective, or $V_1 = 0$, in which case V is a direct sum of copies of S_k, so equals S_k by indecomposability. We see that $\mathcal{R}_k^+(V) = 0$ if and only if $V(j) = 0$ for all $j \neq i$, and $\operatorname{Ker} \xi_k = 0$, which happens if and only if V is non-zero only at k, so that $V \cong S_k$.

When $V \not\cong S_k$ there is a short exact sequence of vector spaces

$$0 \to \mathcal{R}_k^+(V)(k) \to \bigoplus_{\alpha,t(\alpha)=k} V(o(\alpha)) \to V(k) \to 0$$

from which we see that

$$\dim \mathcal{R}_k^+(V)(k) = \left(\sum_{\alpha,t(\alpha)=k} \dim V(o(\alpha)) \right) - \dim V(k).$$

Also $\dim V(j) = \dim \mathcal{R}_k^+(V)(j)$ if $j \neq k$. From this we see that $\boldsymbol{dim}\mathcal{R}_k^-(V) = r_k \boldsymbol{dim}(V)$ because the formula for $r_k \boldsymbol{dim}(V)$ is the same. In constructing $\mathcal{R}_k^- \mathcal{R}_k^+(V)$ we obtain a similar short exact sequence $0 \to \mathcal{R}_k^+(V)(k) \to \bigoplus_{\alpha,t(\alpha)=k} V(o(\alpha)) \to \mathcal{R}_k^- \mathcal{R}_k^+(V)(k) \to 0$ where $\mathcal{R}_k^- \mathcal{R}_k^+(V)(k)$ is defined to be the cokernel of the map on the left. The property of the cokernel gives a natural isomorphism $\mathcal{R}_k^- \mathcal{R}_k^+(V)(k) \to V(k)$ which, together with the identity maps at the other vertices of Q, gives a natural map $\mathcal{R}_k^- \mathcal{R}_k^+(V) \to V$ that is an isomorphism.

The proof of (2) is similar. $\qquad \square$

Definition 2.0.8. A *subcategory* $\mathcal{D}$ of a category $\mathcal{C}$ is a category whose objects are all objects of $\mathcal{C}$, and where for each pair of objects x, y of $\mathcal{D}$ we have $\operatorname{Hom}_{\mathcal{D}}(x, y) \subseteq \operatorname{Hom}_{\mathcal{C}}(x, y)$, with the same rule of composition as $\mathcal{C}$. A *full subcategory* of $\mathcal{C}$ has the stronger property that $\operatorname{Hom}_{\mathcal{D}}(x, y) = \operatorname{Hom}_{\mathcal{C}}(x, y)$ for all objects x, y in $\mathcal{D}$. If k is a vertex of a quiver Q we write $KFQ\text{-mod}\langle k \rangle$ for the full subcategory of $KFQ\text{-mod}$ whose objects do not have any direct summand isomorphic to S_k.

As a replacement for Corollary 1.19, we have the following.

Corollary 2.0.9. *Let k be a sink in Q. Then there is an equivalence of categories KFQ-mod$\langle k \rangle \simeq KFs_kQ$-mod$\langle k \rangle$ given by the inverse equivalences $\mathcal{R}_k^+$ and $\mathcal{R}_k^-$. Thus a representation V of Q with no S_k summand is indecomposable if and only if $\mathcal{R}_k^+ V$ is an indecomposable representation of s_kQ with no S_k summand, giving a bijection between isomorphism classes of indecomposable representations of Q and of s_kQ that are distinct from S_k. Furthermore, $\mathrm{End}_{KFQ}(V) \cong \mathrm{End}_{KFs_kQ}(\mathcal{R}^+ V)$.*

Proof. Notice that if $V \not\cong S_k$ then $\mathcal{R}_k^+(V) \not\cong S_k$ (if k is a sink) and $\mathcal{R}_k^-(V) \not\cong S_k$ (if k is a source). From this it follows that $\mathcal{R}_k^+$ is a functor KFQ-mod$\langle k \rangle \to KFs_kQ$-mod$\langle k \rangle$ and $\mathcal{R}_k^-$ is a functor in the reverse direction. We have seen that the two composites $\mathcal{R}_k^- \mathcal{R}_k^+$ and $\mathcal{R}_k^+ \mathcal{R}_k^-$ are naturally isomorphic to the corresponding identity functors, so we have an equivalence of categories as claimed. We can deduce everything else from this, but to do some of it explicitly: if V is an indecomposable representation of Q other than S_k and $\mathcal{R}_k^+(V) = V_1 \oplus V_2$ were to decompose, then neither V_1 nor V_2 can be S_k because γ_k is injective by construction of $\mathcal{R}_k^+(V)$. Thus $\mathcal{R}_k^- \mathcal{R}_k^+(V) = \mathcal{R}_k^-(V_1) \oplus \mathcal{R}_k^-(V_2)$ with neither summand 0, which is not possible since this representation is isomorphic to V. Hence $\mathcal{R}_k^+(V)$ is indecomposable. By a similar argument with $\mathcal{R}_k^-$, if $\mathcal{R}_k^-(V)$ is indecomposable, so is V. $\qquad\square$

We now do Corollary 1.20 of [DDPW] and define a $(+)$-*admissible* sequence of vertices in Q.

Corollary 2.0.10. *Let $i_1, \ldots, i_{t+1}$ be a $(+)$-admissible sequence of vertices in the quiver Q such that the roots $r_{i_j} \cdots r_{i_t} \alpha_{i_{t+1}}$ are positive for all $1 \le j \le t$.. Then there is an indecomposable representation of Q with dimension vector $\beta = r_{i_1} \cdots r_{i_t} \alpha_{i_{t+1}}$, unique up to isomorphism.*

Proof. We take the indecomposable representation to be $\mathcal{R}_{i_1}^- \cdots \mathcal{R}_{i_{t+1}}^- S_{t+1}$. $\qquad\square$

We define *Coxeter functors, Coxeter transformation*. We do Lemma 1.22 which assumes that if Q is a Dynkin quiver then the corresponding bilinear form is positive definite and $W(Q)$ is finite. Dynkin implies finite could be done by recognizing these groups explicitly as finite groups.

We define *finite representation type* and follow with Gabriel's Theorem 1.23.
The proof divides up as:

- When Q is a Dynkin quiver the isomorphism types of indecomposable representations are characterized by their dimension vectors, which are a subset of the positive real roots. There are only finitely many such roots.

- If Q is not a Dynkin quiver it has a subquiver that is extended Dynkin. The extended Dynkin quivers are shown to have infinite representation type, at least over an infinite field.

- If Q is a Dynkin quiver then for every positive real root there is an indecomposable representation with that dimension vector.

The proof of the last part uses the following lemma.

Lemma 2.0.11. *Let Q be a Dynkin quiver, let β be a positive real root and let i be a vertex of Q. Then $r_i(\beta)$ is either positive or $\beta = \alpha_i$.*

Proof. Because the bilinear form is positive definite we get

$$
\begin{aligned}
0 &\le (\beta \pm \alpha_i, \beta \pm \alpha_i) \\
&= (\beta, \beta) + (\alpha_i, \alpha_i) \pm 2(\beta, \alpha_i) \\
&= 2(2 \pm (\beta, \alpha_i)).
\end{aligned}
$$

Hence $-2 \le (\beta, \alpha_i) \le 2$. Also the form takes values (x, x) in $2\mathbb{Z}$, because $(x, x) = 2\sum_{i \in Q_0} x_i^2 - \sum_{(i,j), i \ne j} x_i x_j$, so $(\beta, \alpha_i) = -2, 0$ or 2. If $(\beta, \alpha_i) = 2$ then $(\beta - \alpha_i, \beta - \alpha_i) = 0$ and consequently $\beta = \alpha_i$. On the other hand, if $(\beta, \alpha_i) \le 2$ then $r_i(\beta) = \beta - (\beta, \alpha_i)\alpha_i > 0$, because $\beta > 0$. $\square$

The approach used in [DDPW] to show that quivers of extended Dynkin type have infinite representation type seems to need K to be an infinite field. There is another approach to this in the literature, due to Tits, using dimension properties of the action of an algebraic group on a variety. It also relies on having an infinite (or possibly algebraically closed) field. Some further argument is needed to show that the infinite field hypothesis can be removed. It would be nice to avoid this; one approach may be to use the reflection functors to construct infinitely many indecomposable representations over any field, at least on an ad hoc basis.

Example 2.0.12. Let $Q = 1 \bullet \overset{\to}{\underset{\to}{}} \bullet 2$ be a quiver with two vertices and two edges between them, as shown. The Cartan matrix is

$$
C_Q = \begin{bmatrix} 2 & -2 \\ -2 & 2 \end{bmatrix}
$$

and the simple reflections $r_i(v) = v - (\alpha_i, v)\alpha_i$ act via matrices

$$
r_1 = \begin{bmatrix} -1 & 2 \\ 0 & 1 \end{bmatrix}, \quad r_2 = \begin{bmatrix} 1 & 0 \\ 2 & -1 \end{bmatrix}.
$$

From this we calculate that $r_2 r_1(\alpha_2) = 2\delta + \alpha_2$ where $\delta = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ spans the kernel of C_Q, called the *radical* of the corresponding bilinear form. The property that $(x, \delta) = 0$ for all x means that δ is fixed by r_1 and r_2, so that $(r_2 r_1)^n(\alpha_2) = 2n\delta + \alpha_2$. From this we see that the coordinate entries of these roots increase, the roots are positive, and for each of them there is an indecomposable representation of Q with that dimension vector. This provides infinitely many non-isomorphic indecomposable representations over any field, because their dimension vectors are distinct.

Class Activity. In the above example, over $\mathbb{Q}$, how many indecomposable representations are there with each of the possible dimension vectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}$? Answers A:0 B:1 C:$\infty$. Can any representations with these dimension vectors be written $\mathcal{R}_1^+(W)$ or $\mathcal{R}_2^-(W)$, for some W?

Chapter 3

Semisimplicity

3.1 Semisimple representations

This section is extracted from section 1.2 of P.J. Webb, A course in finite group representation theory, Cambridge 2016.

We let A be a ring with a 1 and consider its modules. A non-zero A-module V is said to be *simple* or *irreducible* if V has no A-submodules other than 0 and V.

Example 3.1.1. When A is an algebra over a field, every module of dimension 1 is simple. In Example 1.2.3 we have constructed three representations of $\mathbb{R}S_3$, and they are all simple. The trivial and sign representations are simple because they have dimension 1, and the 2-dimensional representation is simple because, visibly, no 1-dimensional subspace is invariant under the group action. We will see in Example **??** that this is a complete list of the simple representations of S_3 over $\mathbb{R}$.

We see immediately that a non-zero module is simple if and only if it is generated by each of its non-zero elements. Furthermore, the simple A-modules are exactly those of the form A/I for some maximal left ideal I of A: every such module is simple, and given a simple module S with a non-zero element $x \in S$ the A-module homomorphism $A \to S$ specified by $a \mapsto ax$ is surjective with kernel a maximal ideal I, so that $S \cong A/I$. Since all simple modules appear inside A in this way, we may deduce that if A is a finite dimensional algebra over a field there are only finitely many isomorphism types of simple modules, these appearing among the composition factors of A when regarded as a module. As a consequence, the simple A-modules are all finite dimensional.

A module that is the direct sum of simple submodules is said to be *semisimple* or *completely reducible*. We saw in Examples 1.2.8 and 1.2.9 two examples of modules, one of which was semisimple and the other of which was not. Every module of finite composition length is somehow built up out of its composition factors, which are simple modules, and we know from the Jordan–Hölder theorem that these composition factors are determined up to isomorphism, although there may be many composition series. The most rudimentary way these composition factors may be fitted together is as a direct sum, giving a semisimple module. In this case the simple summands are the

composition factors of the module and their isomorphism types and multiplicities are uniquely determined. There may, however, be many ways to find simple submodules of a semisimple module so that the module is their direct sum.

We will now relate the property of semisimplicity to the property that appears in Maschke's theorem, namely that every submodule of a module is a direct summand. Our immediate application of this will be an interpretation of Maschke's theorem, but the results have application in greater generality in situations where R is not a field, or when $|G|$ is not invertible in R. To simplify the exposition we have imposed a finiteness condition in the statement of each result, thereby avoiding arguments that use Zorn's lemma. These finiteness conditions can be removed, and we leave the details to Exercise 3.3.13 at the end of this chapter.

In the special case when the ring A is a field and A-modules are vector spaces the next result is familiar from linear algebra.

Lemma 3.1.2. *Let A be a ring with a 1 and suppose that $U = S_1 + \cdots + S_n$ is an A-module that can be written as the sum of finitely many simple modules $S_1, \ldots, S_n$. If V is any submodule of U there is a subset $I = \{i_1, \ldots, i_r\}$ of $\{1, \ldots, n\}$ such that $U = V \oplus S_{i_1} \oplus \cdots S_{i_r}$. In particular,*

(1) V is a direct summand of U, and

(2) (taking $V = 0$), U is the direct sum of some subset of the S_i, and hence is necessarily semisimple.

Proof. Choose a subset I of $\{1, \ldots, n\}$ maximal subject to the condition that the sum $W = V \oplus (\bigoplus_{i \in I} S_i)$ is a direct sum. Note that $I = \emptyset$ has this property, so we are indeed taking a maximal element of a non-empty collection of subsets. We show that $W = U$. If $W \neq U$ then $S_j \nsubseteq W$ for some j. Now $S_j \cap W = 0$, being a proper submodule of S_j, so $S_j + W = S_j \oplus W$ and we obtain a contradiction to the maximality of I. Therefore $W = U$. The consequences (1) and (2) are immediate. $\square$

Proposition 3.1.3. *Let A be a ring with a 1 and let U be an A-module. The following are equivalent.*

(1) U can be expressed as a direct sum of finitely many simple A-submodules.

(2) U can be expressed as a sum of finitely many simple A-submodules.

(3) U has finite composition length and has the property that every submodule of U is a direct summand of U.

When these three conditions hold, every submodule of U and every factor module of U may also be expressed as the direct sum of finitely many simple modules.

Proof. The implication (1) $\Rightarrow$ (2) is immediate and the implications (2) $\Rightarrow$ (1) and (2) $\Rightarrow$ (3) follow from Lemma 3.1.2. To show that (3) $\Rightarrow$ (1) we argue by induction on the composition length of U, and first observe that hypothesis (3) passes to submodules

of U. For if V is a submodule of U and W is a submodule of V then $U = W \oplus X$ for some submodule X, and now $V = W \oplus (X \cap V)$ by the modular law (Exercise 3.3.2 at the end of this chapter). Proceeding with the induction argument, when U has length 1 it is a simple module, and so the induction starts. If U has length greater than 1, it has a submodule V and by condition (3), $U = V \oplus W$ for some submodule W. Now both V and W inherit condition (3) and are of shorter length, so by induction they are direct sums of simple modules and hence so is U.

We have already observed that every submodule of U inherits condition (3), and so satisfies condition (1) also. Every factor module of U has the form U/V for some submodule V of U. If condition (3) holds then $U = V \oplus W$ for some submodule W that we have just observed satisfies condition (1), and hence so does U/V since $U/V \cong W$. $\qquad\square$

We now present a different version of Maschke's theorem. The assertion remains correct if the words 'finite dimensional' are removed from it, but we leave the proof of this to the exercises.

Corollary 3.1.4. *Let F be a field in which $|G|$ is invertible. Then every finite dimensional FG-module is semisimple.*

Proof. This combines Theorem **??** with the equivalence of the statements of Proposition 3.1.3. $\qquad\square$

This result puts us in very good shape if we want to know about the representations of a finite group over a field in which $|G|$ is invertible — for example any field of characteristic zero. To obtain a description of all possible finite dimensional representations we need only describe the simple ones, and then arbitrary ones are direct sums of these.

The following corollaries to Lemma 3.1.2 will be used on many occasions when we are considering modules that are not semisimple.

Corollary 3.1.5. *Let A be a ring with a 1, and let U be an A-module of finite composition length.*

(1) *The sum of all the simple submodules of U is a semisimple module, that is the unique largest semisimple submodule of U.*

(2) *The sum of all submodules of U isomorphic to some given simple module S is a submodule isomorphic to a direct sum of copies of S. It is the unique largest submodule of U with this property.*

Proof. The submodules described can be expressed as the sum of finitely many submodules by the finiteness condition on U. They are the unique largest submodules with their respective properties since they contain all simple submodules (in case (1)), and all submodules isomorphic to S (in case (2)). $\qquad\square$

The largest semisimple submodule of a module U is called the *socle* of U, and is denoted $\text{Soc}(U)$. There is a dual construction called the radical of U, denoted $\text{Rad}\,U$, that we will study in Chapter 6. It is defined to be the intersection of all the maximal submodules of U, and has the property that it is the smallest submodule of U with semisimple quotient.

Corollary 3.1.6. *Let $U = S_1^{a_1} \oplus \cdots \oplus S_r^{a_r}$ be a semisimple module over a ring A with a 1, where the S_i are non-isomorphic simple A-modules and the a_i are their multiplicities as summands of U. Then each submodule $S_i^{a_i}$ is uniquely determined and is characterized as the unique largest submodule of U expressible as a direct sum of copies of S_i.*

Proof. It suffices to show that $S_i^{a_i}$ contains every submodule of U isomorphic to S_i. If T is any non-zero submodule of U not contained in $S_i^{a_i}$ then for some $j \neq i$ its projection to a summand S_j must be non-zero. If we assume that T is simple this projection will be an isomorphism $T \cong S_j$. Thus all simple submodules isomorphic to S_i are contained in the summand $S_i^{a_i}$. $\qquad\square$

3.2 Summary of Chapter 3

- Representations of G over R are the same thing as RG-modules.

- Semisimple modules may be characterized in several different ways. They are modules that are the direct sum of simple modules, or equivalently the sum of simple modules, or equivalently modules for which every submodule is a direct summand.

- If F is a field in which G is invertible, FG-modules are semisimple.

- The sum of all simple submodules of a module is the unique largest semisimple submodule of that module: the socle.

3.3 Exercises for Chapter 3

Exercise 3.3.1. In Example 1.2.8 prove that there are no invariant subspaces other than the ones listed.

Exercise 3.3.2. (The modular law.) Let A be a ring and $U = V \oplus W$ an A-module that is the direct sum of A-modules V and W. Show by example that if X is any submodule of U then it need not be the case that $X = (V \cap X) \oplus (W \cap X)$. Show that if we make the assumption that $V \subseteq X$ then it is true that $X = (V \cap X) \oplus (W \cap X)$.

Exercise 3.3.3. Suppose that ρ is a finite dimensional representation of a finite group G over $\mathbb{C}$. Show that for each $g \in G$ the matrix $\rho(g)$ is diagonalizable.

Exercise 3.3.4. Let $\phi : U \to V$ be a homomorphism of A-modules. Show that $\phi : (\text{Soc}\,U) \subseteq \text{Soc}\,V$, and that if ϕ is an isomorphism then ϕ restricts to an isomorphism $\text{Soc}\,U \to \text{Soc}\,V$.

Exercise 3.3.5. Let $U = S_1 \oplus \cdots \oplus S_r$ be an A-module that is the direct sum of finitely many simple modules $S_1, \ldots, S_r$. Show that if T is any simple submodule of U then $T \cong S_i$ for some i.

Exercise 3.3.6. Let V be an A-module for some ring A and suppose that V is a sum $V = V_1 + \cdots + V_n$ of simple submodules. Assume further that the V_i are pairwise non-isomorphic. Show that the V_i are the only simple submodules of V and that $V = V_1 \oplus \cdots \oplus V_n$ is their direct sum.

Exercise 3.3.7. Let $G = \langle x, y \mid x^2 = y^2 = 1 = [x, y] \rangle$ be the Klein four-group, $R = \mathbb{F}_2$, and consider the two representations ρ_1 and ρ_2 specified on the generators of G by

$$\rho_1(x) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_1(y) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$$\rho_2(x) = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}, \quad \rho_2(y) = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate the socles of these two representations. Show that neither representation is semisimple.

Exercise 3.3.8. Let $G = C_p = \langle x \rangle$ and $R = \mathbb{F}_p$ for some prime $p \geq 3$. Consider the two representations ρ_1 and ρ_2 specified by

$$\rho_1(x) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \rho_2(x) = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Calculate the socles of these two representations and show that neither representation is semisimple. Show that the second representation is nevertheless the direct sum of two non-zero subrepresentations.

Exercise 3.3.9. Let k be an infinite field of characteristic 2, and $G = \langle x, y \rangle \cong C_2 \times C_2$ be the non-cyclic group of order 4. For each $\lambda \in k$ let $\rho_\lambda(x), \rho_\lambda(y)$ be the matrices

$$\rho_\lambda(x) = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \quad \rho_\lambda(y) = \begin{bmatrix} 1 & 0 \\ \lambda & 1 \end{bmatrix}$$

regarded as linear maps $U_\lambda \to U_\lambda$ where U_λ is a k-vector space of dimension 2 with basis $\{e_1, e_2\}$.

(a) Show that ρ_λ defines a representations of G with representation space U_λ.

(b) Find a basis for $\operatorname{Soc} U_\lambda$.

(c) By considering the effect on $\operatorname{Soc} U_\lambda$, show that any kG-module homomorphism $\alpha : U_\lambda \to U_\mu$ has a triangular matrix $\alpha = \begin{bmatrix} a & 0 \\ b & c \end{bmatrix}$ with respect to the given bases.

(d) Show that if $U_\lambda \cong U_\mu$ as kG-modules then $\lambda = \mu$. Deduce that kG has infinitely many non-isomorphic 2-dimensional representations.

Exercise 3.3.10. Let
$$\rho_1 : G \to GL(V)$$
$$\rho_2 : G \to GL(V)$$
be two representations of G on the same R-module V that are injective as homomorphisms. (We say that such a representation is *faithful.*) Consider the three properties

(1) the RG-modules given by ρ_1 and ρ_2 are isomorphic,

(2) the subgroups $\rho_1(G)$ and $\rho_2(G)$ are conjugate in $GL(V)$,

(3) for some automorphism $\alpha \in \mathrm{Aut}(G)$ the representations ρ_1 and $\rho_2\alpha$ are isomorphic.

Show that $(1) \Rightarrow (2)$ and that $(2) \Rightarrow (3)$. Show also that if $\alpha \in \mathrm{Aut}(G)$ is an inner automorphism (i.e. one of the form 'conjugation by g' for some $g \in G$) then ρ_1 and $\rho_1\alpha$ are isomorphic.

Exercise 3.3.11. One form of the Jordan–Zassenhaus theorem asserts that for each n, $GL(n, \mathbb{Z})$ (that is, $\mathrm{Aut}(\mathbb{Z}^n)$) has only finitely many conjugacy classes of subgroups of finite order. Assuming this, show that for each finite group G and each integer n there are only finitely many isomorphism classes of representations of G on $\mathbb{Z}^n$.

Exercise 3.3.12. (a) Using Proposition 3.1.3 show that if A is a ring for which the regular representation ${}_A A$ is semisimple, then every finitely generated A-module is semisimple.

(b) Extend the result of part (a), using Zorn's lemma, to show that if A is a ring for which the regular representation ${}_A A$ is semisimple, then every A-module is semisimple.

Exercise 3.3.13. Let U be a module for a ring A with a 1. Show that the following three statements are equivalent.

(1) U is a direct sum of simple A-submodules.

(2) U is a sum of simple A-submodules.

(3) Every submodule of U is a direct summand of U.

[Use Zorn's lemma to prove a version of Lemma 3.1.2 that has no finiteness hypothesis and then copy Proposition 3.1.3. This deals with all implications except $(3) \Rightarrow (2)$. For that, use the fact that A has a 1 and hence every (left) ideal is contained in a maximal (left) ideal, combined with condition (3), to show that every submodule of U has a simple submodule. Consider the sum of all simple submodules of U and show that it equals U.]

Exercise 3.3.14. Let RG be the group algebra of a finite group G over a commutative ring R with 1. Let S be a simple RG-module and let I be the anihilator in R of S, that is
$$I = \{r \in R \mid rx = 0 \text{ for all } x \in S\}.$$

Show that I is a maximal ideal in R.

[This question requires some familiarity with standard commutative algebra. We conclude from this result that when considering simple RG modules we may reasonably assume that R is a field, since S may naturally be regarded as an $(R/I)G$-module and R/I is a field.]

3.4   Schur's Lemma and Wedderburn's Theorem

This section is Chapter 2 of P.J. Webb, A course in finite group representation theory, Cambridge 2016.

We present the Artin–Wedderburn structure theorem for semisimple algebras and its immediate consequences. This theorem is the ring-theoretic manifestation of the module-theoretic hypothesis of semisimplicity that was introduced in Chapter 1, and it shows that the kind of algebras that can arise when all modules are semisimple is very restricted. The theorem applies to group algebras over a field in which the group order is invertible (as a consequence of Maschke's theorem), but since the result holds in greater generality we will assume we are working with a finite dimensional algebra A over a field k.

Possibly the most important single technique in representation theory is to consider endomorphism rings. It is the main technique of this chapter and we will see it in use throughout this book. The first result is basic, and will be used time and time again.

Theorem 3.4.1 (Schur's Lemma)**.** *Let A be a ring with a 1 and let S_1 and S_2 be simple A-modules. Then $\mathrm{Hom}_A(S_1, S_2) = 0$ unless $S_1 \cong S_2$, in which case the endomorphism ring $\mathrm{End}_A(S_1)$ is a division ring. If A is a finite dimensional algebra over an algebraically closed field k, then every A-module endomorphism of S_1 is multiplication by some scalar. Thus $\mathrm{End}_A(S_1) \cong k$ in this case.*

Proof. Suppose $\theta : S_1 \to S_2$ is a non-zero homomorphism. Then $0 \neq \theta(S_1) \subseteq S_2$, so $\theta(S_1) = S_2$ by simplicity of S_2 and we see that θ is surjective. Thus $\ker \theta \neq S_1$, so $\ker \theta = 0$ by simplicity of S_1, and θ is injective. Therefore θ is invertible, $S_1 \cong S_2$ and $\mathrm{End}_A(S_1)$ is a division ring.

If A is a finite dimensional k-algebra and k is algebraically closed then S_1 is a finite dimensional vector space. Let θ be an A-module endomorphism of S_1 and let λ be an eigenvalue of θ. Now $(\theta - \lambda I) : S_1 \to S_1$ is a singular endomorphism of A-modules, so $\theta - \lambda I = 0$ and $\theta = \lambda I$. $\square$

We have just seen that requiring k to be algebraically closed guarantees that the division rings $\mathrm{End}_A(S)$ are no larger than k, and this is often a significant simplifying condition. In what follows we sometimes make this requirement, also indicating how the results go more generally. At other times requiring k to be algebraically closed is too strong, but we still want k to have the property that $\mathrm{End}_A(S) = k$ for all simple A-modules S. In this case we call k a *splitting field* for the k-algebra A. The theory

of splitting fields will be developed in Chapter 9; for the moment it suffices know that algebraically closed fields are always splitting fields.

The next result is the main tool in recovering the structure of an algebra from its representations. We use the notation A^{op} to denote the *opposite* ring of A, namely the ring that has the same set and the same addition as A, but with a new multiplication $\cdot$ given by $a \cdot b = ba$.

Lemma 3.4.2. *For any ring A with a 1, $\mathrm{End}_A(_AA) \cong A^{\mathrm{op}}$.*

Proof. We prove the result by writing down homomorphisms in both directions that are inverse to each other. The inverse isomorphisms are

$$\phi \mapsto \phi(1)$$
$$(a \mapsto ax) \leftarrow x.$$

There are several things here that need to be checked: that the second assignment does take values in $\mathrm{End}_A(_AA)$, that the morphisms are ring homomorphisms, and that they are mutually inverse. We leave most of this to the reader, observing only that under the first homomorphism a composite $\theta\phi$ is sent to $(\theta\phi)(1) = \theta(\phi(1)) = \theta(\phi(1)1) = \phi(1)\theta(1)$, so that it is indeed a homomorphism to A^{op}. □

Observe that the proof of Lemma 3.4.2 establishes that every endomorphism of the regular representation is of the form 'right multiplication by some element'.

A ring A with 1 all of whose modules are semisimple is itself called *semisimple*. By Exercise 3.3.12 of Chapter 1 it is equivalent to suppose that the regular representation $_AA$ is semisimple. It is also equivalent, if A is a finite dimensional algebra over a field, to suppose that the Jacobson radical of the ring is zero, but the Jacobson radical has not yet been defined and we will not deal with this point of view until Chapter 6.

Theorem 3.4.3 (Artin–Wedderburn)**.** *Let A be a finite dimensional algebra over a field k with the property that every finite dimensional module is semisimple. Then A is a direct sum of matrix algebras over division rings. Specifically, if*

$$_AA \cong S_1^{n_1} \oplus \cdots \oplus S_r^{n_r}$$

where the $S_1, \ldots, S_r$ are non-isomorphic simple modules occuring with multiplicities $n_1, \ldots, n_r$ in the regular representation, then

$$A \cong M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$$

where $D_i = \mathrm{End}_A(S_i)^{\mathrm{op}}$. Furthermore, if k is algebraically closed then $D_i = k$ for all i.

More is true: every such direct sum of matrix algebras is a semisimple algebra. Each matrix algebra over a division ring is a *simple* algebra (namely one that has no 2-sided ideals apart from the zero ideal and the whole ring), and it has up to isomorphism a unique simple module (see the exercises). Furthermore, the matrix algebra summands are uniquely determined as subsets of A (although the module decomposition of $_AA$ is usually only determined up to isomorphism). The uniqueness of the summands will be established in Proposition **??**.

Proof. We first observe that if we have a direct sum decomposition $U = U_1 \oplus \cdots \oplus U_r$ of a module U then $\mathrm{End}_A(U)$ is isomorphic to the algebra of $r \times r$ matrices in which the i, j entries lie in $\mathrm{Hom}_A(U_j, U_i)$. This is because any endomorphism $\phi : U \to U$ may be writen as a matrix of components $\phi = (\phi_{ij})$ where $\phi_{ij} : U_j \to U_i$, and when viewed in this way endomorphisms compose in the manner of matrix multiplication. Since $\mathrm{Hom}_A(S_j^{n_j}, S_i^{n_i}) = 0$ if $i \neq j$ by Schur's lemma, the decomposition of ${}_A A$ shows that

$$\mathrm{End}_A({}_A A) \cong \mathrm{End}_A(S_1^{n_1}) \oplus \cdots \oplus \mathrm{End}_A(S_r^{n_r})$$

and furthermore $\mathrm{End}_A(S_i^{n_i}) \cong M_{n_i}(D_i^{\mathrm{op}})$. Evidently $M_{n_i}(D_i^{\mathrm{op}})^{\mathrm{op}} \cong M_{n_i}(D_i)$ and by Lemma 3.4.2 we identify $\mathrm{End}_A({}_A A)$ as A^{op}. Putting these pieces together gives the matrix algebra decomposition. Finally, if k is algebraically closed it is part of Schur's lemma that $D_i = k$ for all i. $\qquad\square$

Corollary 3.4.4. *Let A be a finite dimensional semisimple algebra over a field k. In any decomposition*

$$ {}_A A = S_1^{n_1} \oplus \cdots \oplus S_r^{n_r} $$

where the S_i are pairwise non-isomorphic simple modules we have that $S_1, \ldots, S_r$ is a complete set of representatives of the isomorphism classes of simple A-modules. When k is algebraically closed $n_i = \dim_k S_i$ and $\dim_k A = n_1^2 + \cdots + n_r^2$.

Proof. All isomorphism types of simple modules must appear in the decomposition because every simple module can be expressed as a homomorphic image of ${}_A A$ (as observed at the start of this chapter), and so must be a homomorphic image of one of the modules S_i. When k is algebraically closed all the division rings D_i coincide with k by Schur's lemma, and $\mathrm{End}_A(S_i^{n_i}) \cong M_{n_i}(k)$. The ring decomposition $A = M_{n_1}(k) \oplus \cdots \oplus M_{n_r}(k)$ of Theorem 3.4.3 immediately gives $\dim_k A = n_1^2 + \cdots + n_r^2$.

We obtained this decomposition by identifying A with $\mathrm{End}({}_A A)^{\mathrm{op}}$ in such a way that an element $a \in A$ is identified with the endomorphism 'right multiplication by a', by Lemma 3.4.2. From this we see that right multiplication of an element of $S_j^{n_j}$ by an element of $M_{n_i}(k)$ is 0 if $i \neq j$, and hence $S_i^{n_i}$ is the unique summand of A (in the initial decomposition of A) containing elements on which $M_{n_i}(k)$ acts in a non-zero fashion from the right. We deduce that $M_{n_i}(k) \cong S_i^{n_i}$ as left A-modules, since the term on the left is isomorphic to the quotient of A by the left submodule consisting of elements that the summand $M_{n_i}(k)$ annihilates by right multiplication, the term on the right is an image of this quotient, and in order to have $\dim_k A = \sum_i \dim_k S_i^{n_i}$ they must be isomorphic. Hence

$$\dim_k M_{n_i}(k) = n_i^2 = \dim_k S_i^{n_i} = n_i \dim S_i,$$

and so $\dim S_i = n_i$. $\qquad\square$

3.5 Summary of Chapter 3

- Endomorphism algebras of simple modules are division rings.

- Semisimple algebras are direct sums of matrix algebras over division rings.

- For a semisimple algebra over an algebraically closed field, the sum of the squares of the degrees of the simple modules equals the dimension of the algebra.

3.6 Exercises for Chapter 3

Exercise 3.6.1. Let A be a finite dimensional semisimple algebra. Show that A has only finitely many isomorphism types of modules in each dimension. [This is not in general true for algebras that are not semisimple: we saw in Chapter 1 Exercise 3.3.9 that $k[C_2 \times C_2]$ has infinitely many non-isomorphic 2-dimensional representations when k is an infinite field of characteristic 2.]

Exercise 3.6.2. Let D be a division ring and n a natural number.
(a) Show that the natural $M_n(D)$-module, consisting of column vectors of length n with entries in D, is a simple module.
(b) Show that $M_n(D)$ is semisimple and has up to isomorphism only one simple module.
(c) Show that every algebra of the form

$$M_{n_1}(D_1) \oplus \cdots \oplus M_{n_r}(D_r)$$

is semisimple.
(d) Show that $M_n(D)$ is a simple ring, namely one in which the only 2-sided ideals are the zero ideal and the whole ring.

Exercise 3.6.3. Show that for any field k we have $M_n(k) \cong M_n(k)^{\mathrm{op}}$, and in general for any division ring D that given any positive integer n, $M_n(D) \cong M_n(D)^{\mathrm{op}}$ if and only if $D \cong D^{\mathrm{op}}$.

Exercise 3.6.4. Let U be a module for a semisimple finite dimensional algebra A. Show that if $\mathrm{End}_A(U)$ is a division ring then U is simple.

Exercise 3.6.5. Prove the following extension of Corollary 3.4.4:

Theorem. *Let A be a finite dimensional semisimple algebra, S a simple A-module and $D = \mathrm{End}_A(S)$. Then S may be regarded as a module over D and the multiplicity of S as a summand of $_A A$ equals $\dim_D S$.*

Exercise 3.6.6. Let k be a field of characteristic 0 and suppose the simple kG-modules are $S_1, \ldots, S_r$ with degrees $d_i = \dim_k S_i$. Show that $\sum_{i=1}^r d_i^2 \geq |G|$ with equality if and only if $\mathrm{End}_{kG}(S_i) = k$ for all i.

Exercise 3.6.7. Using the fact that $M_n(k)$ has a unique simple module up to isomorphism, prove the Noether-Skolem theorem: every algebra automorphism of $M_n(k)$ is inner, i.e. of the form conjugation by some invertible matrix.

Exercise 3.6.8. Let A be a ring with a 1, and let V be an A-module. An element e in any ring is called *idempotent* if and only if $e^2 = e$.

(a) Show that an endomorphism $e : V \to V$ is a projection onto a subspace W if and only if e is idempotent as an element of $\mathrm{End}_A(V)$. (The term *projection* was defined at the start of the proof of Theorem **??**. It is a linear mapping onto a subspace that is the identity on restriction to that subspace.)

(b) Show that direct sum decompositions $V = W_1 \oplus W_2$ as A-modules are in bijection with expressions $1 = e + f$ in $\mathrm{End}_A(V)$, where e and f are idempotent elements with $ef = fe = 0$. (In case $ef = fe = 0$, e and f are called *orthogonal*.)

(c) A non-zero idempotent element e is called *primitive* if it cannot be expressed as a sum of orthogonal idempotent elements in a non-trivial way. Show that $e \in \mathrm{End}_A(V)$ is primitive if and only if $e(V)$ has no (non-trivial) direct sum decomposition. (In this case $e(V)$ is said to be *indecomposable*.)

(d) Suppose that V is semisimple with finitely many simple summands and let $e_1, e_2 \in \mathrm{End}_A(V)$ be idempotent elements. Show that $e_1(V) \cong e_2(V)$ as A-modules if and only if e_1 and e_2 are conjugate by an invertible element of $\mathrm{End}_A(V)$ (i.e. there exists an invertible A-endomorphism $\alpha : V \to V$ such that $e_2 = \alpha e_1 \alpha^{-1}$).

(e) Let k be a field. Show that all primitive idempotent elements in $M_n(k)$ are conjugate under the action of the unit group $GL_n(k)$. Write down explicitly any primitive idempotent element in $M_3(k)$. (It may help to use Exercise 3.6.2.)

Exercise 3.6.9. (We exploit results from a basic algebra course in our suggested approach to this question.) Let G be a cyclic group of order n and k a field.

(a) By considering a homomorphism $k[X] \to kG$ or otherwise, where $k[X]$ is a polynomial ring, show that $kG \cong k[X]/(X^n - 1)$ as rings.

(b) Suppose that the characteristic of k does not divide n. Use the Chinese Remainder Theorem and separability of $X^n - 1$ to show that when kG is expressed as a direct sum of irreducible representations, no two of the summands are isomorphic, and that their degrees are the same as the degrees of the irreducible factors of $X^n - 1$ in $k[X]$. Deduce, as a special case of Corollary **??**, that when k is algebraically closed all irreducible representations of G have degree 1.

(c) When n is prime and $k = \mathbb{Q}$, use irreducibility of $X^{n-1} + X^{n-2} + \cdots X + 1$ to show that G has a simple module S of degree $n - 1$, and that $\mathrm{End}_{kG}(S) \cong \mathbb{Q}(e^{2\pi i/n})$.

(d) When $k = \mathbb{R}$ and n is odd show that G has $\frac{n-1}{2}$ simple representations of degree 2 as well as the trivial representation of degree 1. When $k = \mathbb{R}$ and n is even show that G has $\frac{n-2}{2}$ simple representations of degree 2 as well as two simple representations of degree 1. If S is one of the simple representations of degree 2 show that $\mathrm{End}_{kG}(S) = \mathbb{C}$.

Exercise 3.6.10. Let $\mathbb{H}$ be the algebra of quaternions, that has a basis over $\mathbb{R}$ consisting of elements $1, i, j, k$ and multiplication determined by the relations

$$i^2 = j^2 = k^2 = -1, \ ij = k, \ jk = i, \ ki = j, \ ji = -k, \ kj = -i, \ ik = -j.$$

You may assume that $\mathbb{H}$ is a division ring. The elements $\{\pm 1, \pm i, \pm j, \pm k\}$ under multiplication form the quaternion group Q_8 of order 8, and it acts on $\mathbb{H}$ by left multiplication, so that $\mathbb{H}$ is a 4-dimensional representation of Q_8 over $\mathbb{R}$.

(a) Show that $\text{End}_{\mathbb{R}Q_8}(\mathbb{H}) \cong \mathbb{H}$, and that $\mathbb{H}$ is simple as a representation of Q_8 over $\mathbb{R}$. [Consider the image of $1 \in \mathbb{H}$ under an endomorphism.]

(b) In the decomposition $\mathbb{R}Q_8 = \bigoplus_{i=1}^{t} M_{n_i}(D_i)$ predicted by Corollary **??**, compute the number of summands t, the numbers n_i and the divisions rings D_i. Show that $\mathbb{R}Q_8$ has no simple representation of dimension 2. [Observe that there are four homomorphisms $Q_8 \to \{\pm 1\} \subset \mathbb{R}$ that give four 1-dimensional representations. Show that, together with the representation of dimension 4, we have a complete set of simple representations.]

(c) The span over $\mathbb{R}$ of the elements $1, i \in \mathbb{H}$ is a copy of the field of complex numbers $\mathbb{C}$, so that $\mathbb{H}$ contains $\mathbb{C}$ as a subfield. We may regard $\mathbb{H}$ as a vector space over $\mathbb{C}$ by letting elements of $\mathbb{C}$ act as scalars on $\mathbb{H}$ by multiplication from the *right*. Show that with the action of Q_8 from the left and of $\mathbb{C}$ from the right, $\mathbb{H}$ becomes a left $\mathbb{C}Q_8$-module. With respect to the basis $\{1, j\}$ for $\mathbb{H}$ over $\mathbb{C}$, write down matrices for the action of the elements $i, j \in Q_8$ on $\mathbb{H}$. Show that this 2-dimensional $\mathbb{C}Q_8$-module is simple, and compute its endomorphism ring $\text{End}_{\mathbb{C}Q_8}(\mathbb{H})$.

(d) Show that $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{H} \cong M_2(\mathbb{C})$.

Chapter 4

Structure of projective modules

4.1 Radicals, socles and the augmentation ideal

The following is section 6.3 of P.J. Webb, A course in finite group representation theory, Cambridge 2016.

At this point we examine further the structure of representations that are not semisimple, and we work in the context of modules for a ring A, that is always supposed to have a 1. At the end of Chapter 1 we defined the *socle* of an A-module U to be the sum of all the simple submodules of U, and we showed (at least in the case that U is finite dimensional) that it is the unique largest semisimple submodule of U. We now work with quotients and define a dual concept, the *radical* of U. We work with quotients instead of submodules, and use the fact that if M is a submodule of U, the quotient U/M is simple if and only if M is a maximal submodule of U. We put

$$\operatorname{Rad} U = \bigcap \{M \mid M \text{ is a maximal submodule of } U\}.$$

In our applications U will always be Noetherian, so provided $U \neq 0$ this intersection will be non-empty and hence $\operatorname{Rad} U \neq U$. If U has no maximal submodules (for example, if $U = 0$, or in more general situations than we consider here where U might not be Noetherian) we set $\operatorname{Rad} U = U$.

Lemma 4.1.1. *Let U be a module for a ring A.*

(1) Suppose that $M_1, \ldots, M_n$ are maximal submodules of U. Then there is a subset $I \subseteq \{1, \ldots, n\}$ such that

$$U/(M_1 \cap \cdots \cap M_n) \cong \bigoplus_{i \in I} U/M_i$$

which, in particular, is a semisimple module.

(2) Suppose further that U has the descending chain condition on submodules. Then $U/\operatorname{Rad} U$ is a semisimple module, and $\operatorname{Rad} U$ is the unique smallest submodule of U with this property.

Proof. (1) Let I be a subset of $\{1, \ldots, n\}$ maximal with the property that the quotient homomorphisms $U/(\bigcap_{i \in I} M_i) \to U/M_i$ induce an isomorphism $U/(\bigcap_{i \in I} M_i) \cong \bigoplus_{i \in I} U/M_i$. We show that $\bigcap_{i \in I} M_i = M_1 \cap \cdots \cap M_n$ and argue by contradiction. If it were not the case, there would exist M_j with $\bigcap_{i \in I} M_i \not\subseteq M_j$. Consider the homomorphism

$$f : U \to (\bigoplus_{i \in I} U/M_i) \oplus U/M_j$$

whose components are the quotient homomorphisms $U \to U/M_k$. This has kernel $M_j \cap \bigcap_{i \in I} M_i$, and it will suffice to show that f is surjective, because this will imply that the larger set $I \cup \{j\}$ has the same property as I, thereby contradicting the maximality of I.

To show that f is surjective let $g : U \to U/\bigcap_{i \in I} M_i \oplus U/M_j$ and observe that $(\bigcap_{i \in I} M_i) + M_j = U$ since the left-hand side is strictly larger than M_j, which is maximal in U. Thus if $x \in U$ we can write $x = y + z$ where $y \in \bigcap_{i \in I} M_i$ and $z \in M_j$. Now $g(y) = (0, x + M_j)$ and $g(z) = (x + \bigcap_{i \in I} M_i, 0)$ so that both summands $U/\bigcap_{i \in I} M_i$ and U/M_j are contained in the image of g and g is surjective. Since f is obtained by composing g with the isomorphism that identifies $U/\bigcap_{i \in I} M_i$ with $\bigoplus_{i \in I} U/M_i$, we deduce that f is surjective.

(2) By the assumption that U has the descending chain condition on submodules, $\operatorname{Rad} U$ must be the intersection of finitely many maximal submodules. Therefore $U/\operatorname{Rad} U$ is semisimple by part (1). If V is a submodule such that U/V is semisimple, say $U/V \cong S_1 \oplus \cdots \oplus S_n$ where the S_i are simple modules, let M_i be the kernel of $U \to U/V \xrightarrow{\text{proj.}} S_i$. Then M_i is maximal and $V = M_1 \cap \cdots \cap M_n$. Thus $V \supseteq \operatorname{Rad} U$, and $\operatorname{Rad} U$ is contained in every submodule V for which U/V is semisimple. $\square$

We define the *radical* of a ring A to be the radical of the regular representation $\operatorname{Rad}_A A$ and write simply $\operatorname{Rad} A$. We present some identifications of the radical that are very important theoretically, and also in determining what it is in particular cases.

Proposition 4.1.2. *Let A be a ring. Then,*

(1) $\operatorname{Rad} A = \{a \in A \mid a \cdot S = 0$ for every simple A-module $S\}$, and

(2) $\operatorname{Rad} A$ is a 2-sided ideal of A.

(3) Suppose further that A is a finite dimensional algebra over a field. Then

   (a) $\operatorname{Rad} A$ is the smallest left ideal of A such that $A/\operatorname{Rad} A$ is a semisimple A-module,

   (b) A is semisimple if and only if $\operatorname{Rad} A = 0$,

   (c) $\operatorname{Rad} A$ is nilpotent, and is the largest nilpotent ideal of A.

   (d) $\operatorname{Rad} A$ is the unique ideal U of A with the property that U is nilpotent and A/U is semisimple.

Proof. (1) Given a simple module S and $0 \neq s \in S$, the module homomorphism $_AA \to S$ given by $a \mapsto as$ is surjective and its kernel is a maximal left ideal M_s. Now if $a \in \operatorname{Rad} A$ then $a \in M_s$ for every S and $s \in S$, so $as = 0$ and a annihilates every simple module. Conversely, if $a \cdot S = 0$ for every simple module S and M is a maximal left ideal then A/M is a simple module. Therefore $a \cdot (A/M) = 0$, which means $a \in M$. Hence $a \in \bigcap_{\text{maximal} M} M = \operatorname{Rad} A$.

(2) Being the intersection of left ideals, $\operatorname{Rad} A$ is also a left ideal of A. Suppose that $a \in \operatorname{Rad} A$ and $b \in A$, so $a \cdot S = 0$ for every simple S. Now $a \cdot bS \subseteq a \cdot S = 0$ so ab has the same property that a does.

(3) (a) and (b) are immediate from Lemma 4.1.1. We prove (c). Choose any composition series

$$0 = A_n \subset A_{n-1} \subset \cdots \subset A_1 \subset A_0 = {_AA}$$

of the regular representation. Since each A_i/A_{i+1} is a simple A-module, $\operatorname{Rad} A \cdot A_i \subseteq A_{i+1}$ by part (1). Hence $(\operatorname{Rad} A)^r \cdot A \subseteq A_r$ and $(\operatorname{Rad} A)^n = 0$.

Suppose now that I is a nilpotent ideal of A, say $I^m = 0$, and let S be any simple A-module. Then

$$0 = I^m \cdot S \subseteq I^{m-1} \cdot S \subseteq \cdots \subseteq IS \subseteq S$$

is a chain of A-submodules of S that are either 0 or S since S is simple. There must be some point where $0 = I^r S \neq I^{r-1}S = S$. Then $IS = I \cdot I^{r-1}S = I^r S = 0$, so in fact that point was the very first step. This shows that $I \subseteq \operatorname{Rad} A$ by part (1). Hence $\operatorname{Rad} A$ contains every nilpotent ideal of A, so is the unique largest such ideal.

Finally (d) follows from (a) and (c): these imply that $\operatorname{Rad} A$ has the properties stated in (d); and, conversely, these conditions on an ideal U imply by (a) that $U \supseteq \operatorname{Rad} A$, and by (c) that $U \subseteq \operatorname{Rad} A$. $\square$

Note that if I is a nilpotent ideal of A then it is always true that $I \subseteq \operatorname{Rad}(A)$ without the assumption that A is a finite dimensional algebra. The argument given to prove part 3c of Proposition 4.1.2 shows this.

For any group G and commutative ring R with a 1, the ring homomorphism

$$\epsilon : RG \to R$$
$$g \mapsto 1 \quad \text{for all } g \in G$$

is called the *augmentation map*. As well as being a ring homomorphism it as a homomorphism of RG-modules, in which case it expresses the trivial representation as a homomorphic image of the regular representation. The kernel of ϵ is called the *augmentation ideal*, and is denoted IG. Evidently IG consists of those elements $\sum_{g \in G} a_g g \in RG$ such that $\sum_{g \in G} a_g = 0$. We now show that when k is a field of characteristic p and G is a p-group this construction gives the radical of kG.

Proposition 4.1.3. *Let G be a finite group and R a commutative ring with a 1.*

(1) Let R denote the trivial RG-module. Then $IG = \{x \in RG \mid x \cdot R = 0\}$.

(2) IG is free as an R-module with basis $\{g - 1 \mid 1 \neq g \in G\}$.

(3) If $R = k$ is a field of characteristic p and G is a p-group then $IG = \operatorname{Rad}(kG)$. It follows that IG is nilpotent in this case.

Proof. (1) The augmentation map ϵ is none other than the linear extension to RG of the homomorphism $\rho : G \to GL(1, R)$ that is the trivial representation. Thus each $x \in RG$ acts on R as multiplication by $\epsilon(x)$, and so will act as 0 precisely if $\epsilon(x) = 0$.

(2) The elements $g - 1$ where g ranges through the non-identity elements of G are linearly independent since the elements g are, and they lie in IG. We show that they span IG. Suppose $\sum_{g \in G} a_g g \in IG$, which means that $\sum_{g \in G} a_g = 0 \in R$. Then

$$\sum_{g \in G} a_g g = \sum_{g \in G} a_g g - \sum_{g \in G} a_g 1 = \sum_{1 \neq g \in G} a_g (g - 1)$$

is an expression as a linear combination of elements $g - 1$.

(3) When G is a p-group and $\operatorname{char}(k) = p$ we have seen in Proposition **??** that k is the only simple kG-module. The result follows by part (1) and Proposition 4.1.2. $\square$

Working in the generality of a finite dimensional algebra A again, the radical of A allows us to give a further description of the radical and socle of a module. We present this result for finite dimensional modules, but it is in fact true without this hypothesis. We leave this stronger version to Exercise **??** at the end of this chapter.

Proposition 4.1.4. *Let A be a finite dimensional algebra over a field k, and U a finite dimensional A-module.*

(1) *The following are all descriptions of $\operatorname{Rad} U$:*

    (a) *the intersection of the maximal submodules of U,*

    (b) *the smallest submodule of U with semisimple quotient,*

    (c) *$\operatorname{Rad} A \cdot U$.*

(2) *The following are all descriptions of $\operatorname{Soc} U$:*

    (a) *the sum of the simple submodules of U,*

    (b) *the largest semisimple submodule of U,*

    (c) *$\{u \in U \mid \operatorname{Rad} A \cdot u = 0\}$.*

Proof. Under the hypothesis that U is finitely generated we have seen the equivalence of descriptions (a) and (b) in Lemma 4.1.1 and Corollary 3.1.5. Our arguments below actually work without the hypothesis of finite generation, provided we assume the results of Exercises 3.3.12 and 3.3.13 from Chapter 1. The reader who is satisfied with a proof for finitely generated modules can assume that the equivalence of (a) and (b) has already been proved.

Let us show that the submodule $\operatorname{Rad} A \cdot U$ in (1)(c) satisfies condition (1)(b). Firstly $U/(\operatorname{Rad} A \cdot U)$ is a module for $A/\operatorname{Rad} A$, which is a semisimple algebra. Hence $U/(\operatorname{Rad} A \cdot U)$ is a semisimple module and so $\operatorname{Rad} A \cdot U$ contains the submodule of

(1)(b). On the other hand if $V \subseteq U$ is a submodule for which U/V is semisimple then $\operatorname{Rad} A \cdot (U/V) = 0$ by Proposition 4.1.2, so $V \supseteq \operatorname{Rad} A \cdot U$. In particular, the submodule of (1)(b) contains $\operatorname{Rad} A \cdot U$. This shows that the descriptions in (1)(b) and (1)(c) are equivalent.

To show that they give the same submodule as (1)(a), observe that if V is any maximal submodule of U, then as above (since U/V is simple) $V \supseteq \operatorname{Rad} A \cdot U$, so the intersection of maximal submodules of U contains $\operatorname{Rad} A \cdot U$. The intersection of maximal submodules of the semisimple module $U/(\operatorname{Rad} A \cdot U)$ is zero, so this gives a containment the other way, since they all correspond to maximal submodules of U. We deduce that the intersection of maximal submodules of U equals $\operatorname{Rad} A \cdot U$.

For the conditions in (2), observe that $\{u \in U \mid \operatorname{Rad} A \cdot u = 0\}$ is the largest submodule of U annihilated by $\operatorname{Rad} A$. It is thus an $A/\operatorname{Rad} A$-module and hence is semisimple. Since every semisimple submodule of U is annihilated by $\operatorname{Rad} A$, it equals the largest such submodule. $\qquad\square$

Example 4.1.5. Consider the situation of Theorem **??** and Proposition **??** in which G is a cyclic group of order p^n and k is a field of characteristic p. We see that $\operatorname{Rad} U_r \cong U_{r-1}$ and $\operatorname{Soc} U_r \cong U_1$ for $1 \leq r \leq p^n$, taking $U_0 = 0$.

We now iterate the notions of socle and radical: for each A-module U we define inductively

$$\operatorname{Rad}^n(U) = \operatorname{Rad}(\operatorname{Rad}^{n-1}(U))$$
$$\operatorname{Soc}^n(U)/\operatorname{Soc}^{n-1}(U) = \operatorname{Soc}(U/\operatorname{Soc}^{n-1} U).$$

It is immediate from Proposition 4.1.4 that

$$\operatorname{Rad}^n(U) = (\operatorname{Rad} A)^n \cdot U$$
$$\operatorname{Soc}^n(U) = \{u \in U \mid (\operatorname{Rad} A)^n \cdot u = 0\}$$

and these submodules of U form chains

$$\cdots \subseteq \operatorname{Rad}^2 U \subseteq \operatorname{Rad} U \subseteq U$$
$$0 \subseteq \operatorname{Soc} U \subseteq \operatorname{Soc}^2 U \subseteq \cdots$$

that are called, respectively, the *radical series* and *socle series* of U. The radical series of U is also known as the *Loewy* series of U. The quotients $\operatorname{Rad}^{n-1}(U)/\operatorname{Rad}^n(U)$ are called the *radical layers*, or *Loewy layers* of U, and the quotients $\operatorname{Soc}^n(U)/\operatorname{Soc}^{n-1}(U)$ are called the *socle layers* of U.

The next corollary is a deduction from Proposition 4.1.4, and again it is true without the hypothesis that the modules be finite dimensional.

Corollary 4.1.6. *Let A be a finite dimensional algebra over a field k, and let U and V be finite dimensional A-modules. Then for each n we have $\operatorname{Rad}^n(U \oplus V) = \operatorname{Rad}^n(U) \oplus \operatorname{Rad}^n(V)$ and $\operatorname{Soc}^n(U \oplus V) = \operatorname{Soc}^n(U) \oplus \operatorname{Soc}^n(V)$.*

Proof. One way to see this is to use the identifications $\operatorname{Rad}^n(U \oplus V) = (\operatorname{Rad} A)^n \cdot (U \oplus V)$ and $\operatorname{Soc}^n(U \oplus V) = \{(u, v) \in U \oplus V \mid (\operatorname{Rad} A)^n \cdot (u, v) = 0\}$. $\qquad\square$

The next result can be proved in various ways; it is also a consequence of Theorem 4.4.9 in the next chapter.

Corollary 4.1.7. *Let k be a field of characteristic p and G a p-group. Then the regular representation kG is indecomposable.*

Proof. If $kG = U \oplus V$ is the direct sum of two non-zero modules then $\operatorname{Rad} kG = \operatorname{Rad} U \oplus \operatorname{Rad} V$ where $\operatorname{Rad} U \neq U$ and $\operatorname{Rad} V \neq V$, so the codimension of $\operatorname{Rad} kG$ in kG must be at least 2. We know from Proposition 4.1.3 that $\operatorname{Rad} kG$ has codimension 1, a contradiction. $\square$

Proposition 4.1.8. *Let A be a finite dimensional algebra over a field k, and U an A-module. The radical series of U is the fastest descending series of submodules of U with semisimple quotients, and the socle series of U is the fastest ascending series of U with semisimple quotients. The two series terminate, and if m and n are the least integers for which $\operatorname{Rad}^m U = 0$ and $\operatorname{Soc}^n U = U$ then $m = n$.*

Proof. Suppose that $\cdots \subseteq U_2 \subseteq U_1 \subseteq U_0 = U$ is a series of submodules of U with semisimple quotients. We show by induction on r that $\operatorname{Rad}^r(U) \subseteq U_r$. This is true when $r = 0$. Suppose that $r > 0$ and $\operatorname{Rad}^{r-1}(U) \subseteq U_{r-1}$. Then

$$\operatorname{Rad}^{r-1}(U)/(\operatorname{Rad}^{r-1}(U) \cap U_r) \cong (\operatorname{Rad}^{r-1}(U) + U_r)/U_r \subseteq U_{r-1}/U_r$$

is semisimple, so $\operatorname{Rad}^{r-1}(U) \cap U_r \supseteq \operatorname{Rad}(\operatorname{Rad}^{r-1}(U)) = \operatorname{Rad}^r(U)$. Therefore $\operatorname{Rad}^r(U) \subseteq U_r$. This shows that the radical series descends at least as fast as the series U_i. The argument that the socle series ascends at least as fast is similar.

Since A is a finite dimensional algebra we have $(\operatorname{Rad} A)^r = 0$ for some r. Then $\operatorname{Rad}^r U = (\operatorname{Rad} A)^r \cdot U = 0$ and $\operatorname{Soc}^r U = \{u \in U \mid (\operatorname{Rad} A)^r u = 0\} = U$, so the two series terminate. By what we have just proved, the radical series descends at least as fast as the socle series and so has equal or shorter length. By a similar argument (using the fact that the socle series is the fastest ascending series with semisimple quotients) the socle series ascends at least as fast as the radical series and so has equal or shorter length. We conclude that the two lengths are equal. $\square$

The common length of the radical series and socle series of U is called the *Loewy length* of the module U, and from the description of the terms of these series we see it is the least integer n such that $(\operatorname{Rad} A)^n \cdot U = 0$.

4.2 Characterizations of projective and injective modules

The following is chapter 7 of P.J. Webb, A course in finite group representation theory, Cambridge 2016.

Recall that a module M over a ring A is said to be *free* if it has a basis; that is, a subset $\{x_i \mid i \in I\}$ that spans M as an A-module, and is linearly independent over A. To say that $\{x_i \mid i \in I\}$ is a basis of M is equivalent to requiring $M = \bigoplus_{i \in I} A x_i$ with $A \cong A x_i$ via an isomorphism $a \mapsto a x_i$ for all i. Thus M is a finitely generated

free module if and only if $M \cong A^n$ for some n. These conditions are also equivalent to the condition in the following proposition:

Proposition 4.2.1. *Let A be a ring and M an A-module. The following are equivalent for a subset $\{x_i \mid i \in I\}$ of M:*

(1) *$\{x_i \mid i \in I\}$ is a basis of M,*

(2) *for every module N and mapping of sets $\phi : \{x_i \mid i \in I\} \to N$ there exists a unique module homomorphism $\psi : M \to N$ that extends ϕ.*

Proof. The proof is standard. If $\{x_i \mid i \in I\}$ is a basis, then given ϕ we may define $\psi(\sum_{i \in I} a_i x_i) = \sum_{i \in I} a_i \phi(x_i)$ and this is evidently the unique module homomorphism extending ϕ. This shows that (1) implies (2). Conversely if condition (2) holds we may construct the free module F with $\{x_i \mid i \in I\}$ as a basis and use the condition to construct a homomorphism from $M \to F$ that is the identity on $\{x_i \mid i \in I\}$. The fact just shown that the free module also satisfies condition (2) allows us to construct a homomorphism $F \to M$ that is again the identity on $\{x_i \mid i \in I\}$, and the two homomorphisms have composites in both directions that are the identity, since these are the unique extensions of the identity map on $\{x_i \mid i \in I\}$. They are therefore isomorphisms and from this condition (1) follows. $\square$

We define a module homomorphism $f : M \to N$ to be a *split epimorphism* if and only if there exists a homomorphism $g : N \to M$ so that $fg = 1_N$, the identity map on N. Note that a split epimorphism is necessarily an epimorphism since if $x \in N$ then $x = f(g(x))$ so that x lies in the image of f. We define similarly f to be a *split monomorphism* if there exists a homomorphism $g : N \to M$ so that $gf = 1_M$. Necessarily a split monomorphism is a monomorphism. We are about to show that if f is a split epimorphism then N is (isomorphic to) a direct summand of M. To combine both this and the corresponding result for split monomorphisms it is convenient to introduce short exact sequences. We say that a diagram of modules and module homomorphisms $L \xrightarrow{\alpha} M \xrightarrow{\beta} N$ is *exact* at M if $\operatorname{Im} \alpha = \ker \beta$. A *short exact sequence* of modules is a diagram $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ that is exact at each of L, M and N. Exactness at L and N means simply that α is a monomorphism and β is an epimorphism.

Proposition 4.2.2. *Let $0 \to L \xrightarrow{\alpha} M \xrightarrow{\beta} N \to 0$ be a short exact sequence of modules over a ring. The following are equivalent:*

(1) *α is a split monomorphism,*

(2) *β is a split epimorphism,*

(3) there is a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & L & \xrightarrow{\alpha} & M & \xrightarrow{\beta} & N & \to & 0 \\
& & \| & & \downarrow{\gamma} & & \| & & \\
0 & \to & L & \xrightarrow{\iota_1} & L \oplus N & \xrightarrow{\pi_2} & N & \to & 0
\end{array}
$$

where ι_1 and π_2 are inclusion into the first summand and projection onto the second summand,

(4) for every module U the sequence

$$0 \to \mathrm{Hom}_A(U, L) \to \mathrm{Hom}_A(U, M) \to \mathrm{Hom}_A(U, N) \to 0$$

is exact,

(4′) for every module U the sequence

$$0 \to \mathrm{Hom}_A(N, U) \to \mathrm{Hom}_A(M, U) \to \mathrm{Hom}_A(L, U) \to 0$$

is exact.

In any diagram such as the one in (3) the morphism γ is necessarily an isomorphism. Thus if any of the listed conditions is satisfied it follows that $M \cong L \oplus N$.

Proof. Condition (3) implies the first two, since the existence of such a commutative diagram implies that α is split by $\pi_1\gamma$ and β is split by $\gamma^{-1}\iota_2$, and it also implies the last two conditions because the commutative diagram produces similar commutative diagrams after applying $\mathrm{Hom}_A(U, -)$ and $\mathrm{Hom}_A(-, U)$.

Conversely if condition (1) is satisfied, so that $\delta\alpha = 1_L$ for some homomorphism $\delta : M \to L$, we obtain a commutative diagram as in (3) on taking the components of γ to be δ and β. If condition (2) is satisfied we obtain a commutative diagram similar to the one in (3) but with a homomorphism $\zeta : L \oplus N \to M$ in the wrong direction, whose components are α and a splitting of β. We obtain the diagram of (3) on showing that in any such diagram the middle vertical homomorphism must be invertible.

The fact that the middle homomorphism in the diagram must be invertible is a consequence of both the 'five lemma' and the 'snake lemma' in homological algebra. We leave it here as an exercise.

Finally if (4) holds then on taking U to be N we deduce that the identity map on N is the image of a homomorphism $\epsilon : U \to M$, so that $1_N = \beta\epsilon$ and β is split epi, so that (2) holds. Equally if (4′) holds then taking U to be L we see that the identity map on L is the image of a homomorphism $\delta : M \to U$, so that $1_L = \delta\alpha$ and (1) holds. □

In the event that α and β are split, we say that the short exact sequence in Proposition 4.2.2 is *split*. Notice that whenever $\beta : M \to N$ is an epimorphism it is part of the short exact sequence $0 \to \ker\beta \hookrightarrow M \xrightarrow{\beta} N \to 0$, and so we deduce that if β is a split epimorphism then N is a direct summand of M. A similar comment evidently applies to split monomorphisms.

Proposition 4.2.3. *The following are equivalent for an A-module P.*

(1) *P is a direct summand of a free module.*

(2) *Every epimorphism $V \to P$ is split.*

(3) *For every pair of morphisms*

$$P$$
$$\downarrow \alpha$$
$$V \xrightarrow{\beta} W$$

where β is an epimorphism, there exists a morphism $\gamma : P \to V$ with $\beta\gamma = \alpha$.

(4) *For every short exact sequence of A-modules $0 \to V \to W \to X \to 0$ the corresponding sequence*

$$0 \to \operatorname{Hom}_A(P, V) \to \operatorname{Hom}_A(P, W) \to \operatorname{Hom}_A(P, X) \to 0$$

is exact.

Proof. This result is standard and we do not prove it here. In condition (4) the sequence of homomorphism groups is always exact at the left-hand terms $\operatorname{Hom}_A(P, V)$ and $\operatorname{Hom}_A(P, W)$ without requiring any special property of P (we say that $\operatorname{Hom}_A(P,)$ is *left exact*). The force of condition (4) is that the sequence should be exact at the right-hand term. □

We say that a module P satisfying any of the four conditions of Proposition 4.2.3 is *projective*. Notice that direct sums and also direct summands of projective modules are projective. An indecomposable module that is projective is an indecomposable projective module, and these modules will be very important in our study. In other texts the indecomposable projective modules are also known as PIMs, or Principal Indecomposable Modules, but we will not use this terminology here.

We should also mention injective modules, which enjoy properties similar to those of projective modules, but in a dual form. We say that a module I is *injective* if and only if whenever there are morphisms

$$I$$
$$\uparrow \alpha$$
$$V \xleftarrow{\beta} W$$

with β a monomorphism, then there exists a morphism $\gamma : V \to I$ so that $\gamma\beta = \alpha$. Dually to Proposition 4.2.3, it is equivalent to require that every monomorphism $I \to V$ is split; and also that $\operatorname{Hom}_A(\_, I)$ sends exact sequences to exact sequences. When A is an arbitrary ring we do not have such a nice characterization of injectives analogous to the property that projective modules are direct summands of free modules. However, for group algebras over a field we will show in Corollary 8.13 that injective modules are the same thing as projective modules, so that in this context they are indeed summands of free modules.

4.3   Projectives by means of idempotents

One way to obtain projective A-modules is from idempotents of the ring A. If $e^2 = e \in A$ then $_AA = Ae \oplus A(1-e)$ as A-modules, and so the submodules Ae and $A(1-e)$ are projective. We formalize this with the next result, which should be compared with Proposition **??** in which we were dealing with ring summands of A and central idempotents.

Proposition 4.3.1. *Let A be a ring. The decompositions of the regular representation as a direct sum of submodules*

$$_AA = A_1 \oplus \cdots \oplus A_r$$

biject with expressions $1 = e_1 + \cdots + e_r$ for the identity of A as a sum of orthogonal idempotents, in such a way that $A_i = Ae_i$. The summand A_i is indecomposable if and only if the idempotent e_i is primitive.

Proof. Suppose that $1 = e_1 + \cdots + e_r$ is an expression for the identity as a sum of orthogonal idempotents. Then

$$_AA = Ae_1 \oplus \cdots \oplus Ae_r,$$

for the Ae_i are evidently submodules of A, and their sum is A since if $x \in A$ then $x = xe_1 + \cdots + xe_r$. The sum is direct since if $x \in Ae_i \cap \sum_{j \neq i} Ae_j$ then $x = xe_i$ and also $x = \sum_{j \neq i} a_j e_j$ so $x = xe_i = \sum_{j \neq i} a_j e_j e_i = 0$.

Conversely, suppose that $_AA = A_1 \oplus \cdots \oplus A_r$ is a direct sum of submodules. We may write $1 = e_1 + \cdots + e_r$ where $e_i \in A_i$ is a uniquely determined element. Now $e_i = e_i 1 = e_i e_1 + \cdots + e_i e_r$ is an expression in which $e_i e_j \in A_j$, and since the only such expression is e_i itself we deduce that

$$e_i e_j = \begin{cases} e_i & \text{if } i = j, \\ 0 & \text{otherwise.} \end{cases}$$

The two constructions just described, in which we associate an expression for 1 as a sum of idempotents to a module direct sum decomposition and vice-versa, are mutually inverse, giving a bijection as claimed.

If a summand A_i decomposes as the direct sum of two other summands, this gives rise to an expression for e_i as a sum of two orthogonal idempotents, and conversely. Thus A_i is indecomposable if and only if e_i is primitive. $\square$

In Proposition **??** it was proved that in a decomposition of A as a direct sum of indecomposable rings, the rings are uniquely determined as subsets of A and the corresponding primitive central idempotents are also unique. We point out that the corresponding uniqueness property need not hold with module decompositions of $_AA$

that are not ring decompositions. For an example of this we take $A = M_2(R)$, the ring of 2×2-matrices over a ring R, and consider the two decompositions

$$_A A = A \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \oplus A \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = A \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix} \oplus A \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

The submodules here are all different. We will see later that if A is a finite dimensional algebra over a field then in any two decompositions of $_A A$ as a direct sum of indecomposable submodules, the submodules are isomorphic in pairs.

We will also see that when A is a finite dimensional algebra over a field, every indecomposable projective A-module may be realized as Ae for some primitive idempotent e. For other rings this need not be true: an example is $\mathbb{Z}G$, for which it is the case that the only idempotents are 0 and 1 (see Exercise **??** in Chapter 8). For certain finite groups (an example is the cyclic group of order 23, but this takes us beyond the scope of this book) there exist indecomposable projective $\mathbb{Z}G$-modules that are not free, so such modules will never have the form $\mathbb{Z}Ge$ for any idempotent element e.

Example 4.3.2. We present an example of a decomposition of the regular representation in a situation that is not semisimple. Many of the observations we will make are consequences of theory to be presented in later sections, but it seems worthwhile to show that the calculations can be done by direct arguments.

Consider the group ring $\mathbb{F}_4 S_3$ where $\mathbb{F}_4$ is the field of 4 elements. The choice of $\mathbb{F}_4$ is made because at one point it will be useful to have all cube roots of unity available, but in fact many of the observations we are about to make also hold over the field $\mathbb{F}_2$. By Proposition **??** the 1-dimensional representations of S_3 are the simple representations of $S_3/S_3' \cong C_2$, lifted to S_3. But $\mathbb{F}_4 C_2$ has only one simple module, namely the trivial module, by Proposition **??**, so this is the only 1-dimensional $\mathbb{F}_4 S_3$-module. The 2-dimensional representation of S_3 constructed in Chapter 1 over any coefficient ring is now seen to be simple here, since otherwise it would have a trivial submodule; but the eigenvalues of the element $(1,2,3)$ on this module are ω and ω^2, where $\omega \in \mathbb{F}_4$ is a primitive cube root of 1, so there is no trivial submodule.

Let $K = \langle (1,2,3) \rangle$ be the subgroup of S_3 of order 3. Now $\mathbb{F}_4 K$ is semisimple with three 1-dimensional representations on which $(1,2,3)$ acts as 1, ω and ω^2, respectively. In fact

$$\mathbb{F}_4 K = \mathbb{F}_4 K e_1 \oplus \mathbb{F}_4 K e_2 \oplus \mathbb{F}_4 K e_3$$

where

$$e_1 = () + (1,2,3) + (1,3,2)$$
$$e_2 = () + \omega(1,2,3) + \omega^2(1,3,2)$$
$$e_3 = () + \omega^2(1,2,3) + \omega(1,3,2)$$

are orthogonal idempotents in $\mathbb{F}_4 K$. We may see that these are orthogonal idempotents by direct calculation, but it can also be seen by observing that the corresponding elements of $\mathbb{C}K$ with ω replaced by $e^{\frac{2\pi i}{3}}$ are orthogonal and square to 3 times themselves (Theorem **??**), and lie in $\mathbb{Z}[e^{\frac{2\pi i}{3}}]K$. Reduction modulo 2 gives a ring homomorphism

$\mathbb{Z}[e^{\frac{2\pi i}{3}}] \to \mathbb{F}_4$ that maps these elements to e_1, e_2 and e_3, while retaining their properties. Thus

$$\mathbb{F}_4 S_3 = \mathbb{F}_4 S_3 e_1 \oplus \mathbb{F}_4 S_3 e_2 \oplus \mathbb{F}_4 S_3 e_3$$

and we have constructed modules $\mathbb{F}_4 S_3 e_i$ that are projective. We have not yet shown that they are indecomposable.

We easily compute that

$$(1,2,3)e_1 = e_1, \quad (1,2,3)e_2 = \omega^2 e_2, \quad (1,2,3)e_3 = \omega e_3$$

and from this we see that $K \cdot \mathbb{F}_4 e_i = \mathbb{F}_4 e_i$ for all i. Since $S_3 = K \cup (1,2)K$ we have $\mathbb{F}_4 S_3 e_i = \mathbb{F}_4 e_i \oplus \mathbb{F}_4 (1,2)e_i$, which has dimension 2 for all i. We have already seen that when $i = 2$ or 3, e_i is an eigenvector for $(1,2,3)$ with eigenvalue ω or ω^2, and a similar calculation shows that the same is true for $(1,2)e_i$. Thus when $i = 2$ or 3, $\mathbb{F}_4 S_3 e_i$ has no trivial submodule and hence is simple by the observations made at the start of this example. We have an isomorphism of $\mathbb{F}_4 S_3$-modules

$$\mathbb{F}_4 S_3 e_2 \to \mathbb{F}_4 S_3 e_3$$
$$e_2 \mapsto (1,2)e_3$$
$$(1,2)e_2 \mapsto e_3.$$

On the other hand $\mathbb{F}_4 S_3 e_1$ has fixed points $\mathbb{F}_4 \sum_{g \in S_3} g$ of dimension 1 and so has two composition factors, which are trivial. On restriction to $\mathbb{F}_4 \langle (1,2) \rangle$ it is the regular representation, and it is a uniserial module.

We see from all this that $\mathbb{F}_4 S_3 = \begin{smallmatrix} 1 \\ 1 \end{smallmatrix} \oplus 2 \oplus 2$, in a diagrammatic notation. Thus the 2-dimensional simple $\mathbb{F}_4 S_3$-module is projective, and the trivial module appears as the unique simple quotient of a projective module of dimension 2 whose socle is also the trivial module. These summands of $\mathbb{F}_4 S_3$ are indecomposable, and so e_1, e_2 and e_3 are primitive idempotents in $\mathbb{F}_4 S_3$. We see also that the radical of $\mathbb{F}_4 S_3$ is the span of $\sum_{g \in S_3} g$.

4.4 Projective covers, Nakayama's lemma and lifting of idempotents

We now develop the theory of projective covers. We first make the definition that an *essential epimorphism* is an epimorphism of modules $f : U \to V$ with the property that no proper submodule of U is mapped surjectively onto V by f. An equivalent formulation is that whenever $g : W \to U$ is a map such that fg is an epimorphism, then g is an epimorphism. One immediately asks for examples of essential epimorphisms, but it is probably more instructive to consider epimorphisms that are not essential. If $U \to V$ is any epimorphism and X is a non-zero module then the epimorphism $U \oplus X \to V$ constructed as the given map on U and zero on X can never be essential. This is because U is a submodule of $U \oplus X$ mapped surjectively onto V. Thus if $U \to V$ is essential then U can have no direct summands that are mapped to zero. One may

think of an essential epimorphism as being minimal, in that no unnecessary parts of U are present.

The greatest source of essential epimorphisms is Nakayama's lemma, given here in a version for modules over non-commutative rings. Over an arbitrary ring a finiteness condition is required, and that is how we state the result here. We will see in Exercise 4.8.10 at the end of this chapter that, when the ring is a finite dimensional algebra over a field, the result is true for arbitrary modules without any finiteness condition.

Theorem 4.4.1 (Nakayama's Lemma). *If U is any Noetherian module, the homomorphism $U \to U/\operatorname{Rad} U$ is essential. Equivalently, if V is a submodule of U with the property that $V + \operatorname{Rad} U = U$, then $V = U$.*

Proof. Suppose V is a submodule of U. If $V \neq U$ then $V \subseteq M \subset U$ where M is a maximal submodule of U. Now $V + \operatorname{Rad} U \subseteq M$ and so the composite $V \to U \to U/\operatorname{Rad} U$ has image contained in $M/\operatorname{Rad} U$, which is not equal to $U/\operatorname{Rad} U$ since $(U/\operatorname{Rad} U)/(M/\operatorname{Rad} U) \cong U/M \neq 0$. $\qquad\square$

When U is a module for a finite dimensional algebra it is always true that every proper submodule of U is contained in a maximal submodule, even when U is not finitely generated. This was the only point in the proof of Theorem 4.4.1 where the Noetherian hypothesis was used, and so in this situation $U \to U/\operatorname{Rad} U$ is always essential. This is shown in Exercise 4.8.10 of this chapter.

The next result is not at all difficult and could also be proved as an exercise.

Proposition 4.4.2.   *(1) Suppose that $f : U \to V$ and $g : V \to W$ are two module homomorphsms. If two of f, g and gf are essential epimorphisms then so is the third.*

(2) Let $f : U \to V$ be a homomorphism of Noetherian modules. Then f is an essential epimorphism if and only if the homomorphism of radical quotients $U/\operatorname{Rad} U \to V/\operatorname{Rad} V$ is an isomorphism.

(3) Let $f_i : U_i \to V_i$ be homomorphisms of Noetherian modules, where $i = 1, \ldots, n$. The f_i are all essential epimorphisms if and only if

$$\oplus f_i : \bigoplus_i U_i \to \bigoplus_i V_i$$

is an essential epimorphism.

Proof. (1) Suppose f and g are essential epimorphisms. Then gf is an epimorphism also, and it is essential because if U_0 is a proper submodule of U then $f(U_0)$ is a proper submodule of V since f is essential, and hence $g(f(U_0))$ is a proper submodule of S since g is essential.

Next suppose f and gf are essential epimorphisms. Since $W = \operatorname{Im}(gf) \subseteq \operatorname{Im}(g)$ it follows that g is an epimorphism. If V_0 is a proper submodule of V then $f^{-1}(V_0)$ is a

proper submodule of U since f is an epimorphism, and now $g(V_0) = gf(f^{-1}(V_0))$ is a proper submodule of S since gf is essential.

Suppose that g and gf are essential epimorphisms. If f were not an epimorphism then $f(U)$ would be a proper submodule of V, so $gf(U)$ would be a proper submodule of W since gf is essential. Since $gf(U) = W$ we conclude that f is an epimorphism. If U_0 is a proper submodule of U then $gf(U_0)$ is a proper submodule of W, since gf is essential, so $f(U_0)$ is a proper submodule of V since g is an epimorphism. Hence f is essential.

(2) Consider the commutative square

$$
\begin{array}{ccc}
U & \longrightarrow & V \\
\downarrow & & \downarrow \\
U/\operatorname{Rad} U & \longrightarrow & V/\operatorname{Rad} V
\end{array}
$$

where the vertical homomorphisms are essential epimorphisms by Nakayama's lemma. Now if either of the horizontal arrows is an essential epimorphism then so is the other, using part (1). The bottom arrow is an essential epimorphism if and only if it is an isomorphism; for $U/\operatorname{Rad} U$ is a semisimple module and so the kernel of the map to $V/\operatorname{Rad} V$ has a direct complement in $U/\operatorname{Rad} U$, which maps onto $V/\operatorname{Rad} V$. Thus if $U/\operatorname{Rad} U \to V/\operatorname{Rad} V$ is an essential epimorphism its kernel must be zero and hence it must be an isomorphism.

(3) The map

$$(\oplus_i U_i)/\operatorname{Rad}(\oplus_i U_i) \to (\oplus_i V_i)/\operatorname{Rad}(\oplus_i V_i)$$

induced by $\oplus f_i$ may be identified as a map

$$\bigoplus_i (U_i/\operatorname{Rad} U_i) \to \bigoplus_i (V_i/\operatorname{Rad} V_i),$$

and it is an isomorphism if and only if each map $U_i/\operatorname{Rad} U_i \to V_i/\operatorname{Rad} V_i$ is an isomorphism. These conditions hold if and only if $\oplus f_i$ is an essential epimorphism, if and only if each f_i is an essential epimorphism by part (2). $\qquad\square$

We define a *projective cover* of a module U to be an essential epimorphism $P \to U$, where P is a projective module. Strictly speaking the projective cover is the homomorphism, but we may also refer to the module P as the projective cover of U. We are justified in calling it *the* projective cover by the second part of the following result, which says that projective covers (if they exist) are unique.

Proposition 4.4.3. *(1) Suppose that $f : P \to U$ is a projective cover of a module U and $g : Q \to U$ is an epimorphism where Q is a projective module. Then we may write $Q = Q_1 \oplus Q_2$ so that g has components $g = (g_1, 0)$ with respect to this direct sum decomposition and $g_1 : Q_1 \to U$ appears in a commutative triangle*

$$
\begin{array}{ccc}
 & Q_1 & \\
{\scriptstyle \gamma}\swarrow & & \downarrow{\scriptstyle g_1} \\
P & \xrightarrow{\ f\ } & U
\end{array}
$$

where γ is an isomorphism.

(2) *If any exist, the projective covers of a module U are all isomorphic, by isomorphisms that commute with the essential epimorphisms.*

Proof. (1) In the diagram

$$
\begin{array}{ccc}
 & & Q \\
 & & \downarrow g \\
P & \xrightarrow{f} & U
\end{array}
$$

we may lift in both directions to obtain maps $\alpha : P \to Q$ and $\beta : Q \to P$ so that the two triangles commute. Now $f\beta\alpha = g\alpha = f$ is an epimorphism, so $\beta\alpha$ is also an epimorphism since f is essential. Thus β is an epimorphism. Since P is projective β splits and $Q = Q_1 \oplus Q_2$ where $Q_2 = \ker\beta$, and β maps Q_1 isomorphically to P. Thus $g = (f\beta|_{Q_1}, 0)$ is as claimed with $\gamma = \beta|_{Q_1}$.

(2) Supposing that $f : P \to U$ and $g : Q \to U$ are both projective covers, since Q_1 is a submodule of Q that maps onto U and f is essential we deduce that $Q = Q_1$. Now $\gamma : Q \to P$ is the required isomorphism. $\square$

Corollary 4.4.4. *If P and Q are Noetherian projective modules over a ring then $P \cong Q$ if and only if $P/\operatorname{Rad} P \cong Q/\operatorname{Rad} Q$.*

Proof. By Nakayama's lemma P and Q are the projective covers of $P/\operatorname{Rad} P$ and $Q/\operatorname{Rad} Q$. It is clear that if P and Q are isomorphic then so are $P/\operatorname{Rad} P$ and $Q/\operatorname{Rad} Q$, and conversely if these quotients are isomorphic then so are their projective covers, by uniqueness of projective covers. $\square$

If P is a projective module for a finite dimensional algebra A then Corollary 4.4.4 says that P is determined up to isomorphism by its semisimple quotient $P/\operatorname{Rad} P$. We are going to see that if P is an indecomposable projective A-module, then its radical quotient is simple, and also that every simple A-module arises in this way. Furthermore, every indecomposable projective for a finite dimensional algebra is isomorphic to a summand of the regular representation (something that is not true in general for projective $\mathbb{Z}G$-modules, for instance). This means that it is isomorphic to a module Af for some primitive idempotent $f \in A$, and the radical quotient $P/\operatorname{Rad} P$ is isomorphic to $(A/\operatorname{Rad} A)e$ where e is a primitive idempotent of $A/\operatorname{Rad} A$ satisfying $e = f + \operatorname{Rad} A$. We will examine this kind of relationship between idempotent elements more closely.

In general if I is an ideal of a ring A and f is an idempotent of A then clearly $e = f + I$ is an idempotent of A/I, and we say that f *lifts* e. On the other hand, given an idempotent e of A/I it may or may not be possible to lift it to an idempotent of A. If, for every idempotent e in A/I, we can always find an idempotent $f \in A$ such that $e = f + I$ then we say we can *lift idempotents* from A/I to A.

We present the next results about lifting idempotents in the context of a ring with a nilpotent ideal I, but readers familiar with completions will recognize that these results extend to a situation where A is complete with respect to the I-adic topology on A.

Theorem 4.4.5. *Let I be a nilpotent ideal of a ring A and e an idempotent in A/I. Then there exists an idempotent $f \in A$ with $e = f + I$. If e is primitive, so is any lift f.*

Proof. We define idempotents $e_i \in A/I^i$ inductively such that $e_i + I^{i-1}/I^i = e_{i-1}$ for all i, starting with $e_1 = e$. Suppose that e_{i-1} is an idempotent of A/I^{i-1}. Pick any element $a \in A/I^i$ mapping onto e_{i-1}, so that $a^2 - a \in I^{i-1}/I^i$. Since $(I^{i-1})^2 \subseteq I^i$ we have $(a^2 - a)^2 = 0 \in A/I^i$. Put $e_i = 3a^2 - 2a^3$. This does map to $e_{i-1} \in A/I^{i-1}$ and we have

$$\begin{aligned} e_i^2 - e_i &= (3a^2 - 2a^3)(3a^2 - 2a^3 - 1) \\ &= -(3 - 2a)(1 + 2a)(a^2 - a)^2 \\ &= 0. \end{aligned}$$

This completes the inductive definition, and if $I^r = 0$ we put $f = e_r$.

Suppose that e is primitive and that f can be written $f = f_1 + f_2$ where f_1 and f_2 are orthogonal idempotents. Then $e = e_1 + e_2$, where $e_i = f_i + I$, is also a sum of orthogonal idempotents. Therefore one of these is zero, say, $e_1 = 0 \in A/I$. This means that $f_1^2 = f_1 \in I$. But I is nilpotent, and so contains no non-zero idempotent. $\square$

We will very soon see that in the situation of Theorem 4.4.5, if f is primitive, so is e. It depends on the next result, which is a more elaborate version of Theorem 4.4.5.

Corollary 4.4.6. *Let I be a nilpotent ideal of a ring A and let $1 = e_1 + \cdots + e_n$ be a sum of orthogonal idempotents in A/I. Then we can write $1 = f_1 + \cdots + f_n$ in A, where the f_i are orthogonal idempotents such that $f_i + I = e_i$ for all i. If the e_i are primitive then so are the f_i.*

Proof. We proceed by induction on n, the induction starting when $n = 1$. Suppose that $n > 1$ and the result holds for smaller values of n. We will write $1 = e_1 + E$ in A/I where $E = e_2 + \cdots + e_n$ is an idempotent orthogonal to e_1. By Theorem 4.4.5 we may lift e_1 to an idempotent $f_1 \in A$. Write $F = 1 - f_1$, so that F is an idempotent that lifts E. Now F is the identity element of the ring FAF which has a nilpotent ideal FIF. The composite homomorphism $FAF \hookrightarrow A \to A/I$ has kernel $FAF \cap I$ and this equals FIF, since clearly $FAF \cap I \supseteq FIF$, and if $x \in FAF \cap I$ then $x = FxF \in FIF$, so $FAF \cap I \subseteq FIF$. Inclusion of FAF in A thus induces a monomorphism $FAF/FIF \to A/I$, and its image is $E(A/I)E$. In $E(A/I)E$ the identity element E is the sum of $n - 1$ orthogonal idempotents, and this expression is the image of a similar expression for $F + FIF$ in FAF/FIF. By induction, there is a sum of orthogonal idempotents $F = f_2 + \cdots + f_n$ in FAF that lifts the expression in FAF/FIF and hence also lifts the expression for E in A/I, so we have idempotents $f_i \in A$, $i = 1, \ldots, n$ with $f_i + I = e_i$. These f_i are orthogonal: for $f_2, \ldots, f_n$ are orthogonal in FAF by induction, and if $i > 1$ then $Ff_i = f_i$ so we have $f_1 f_i = f_1 F f_i = 0$.

The final assertion about primitivity is the last part of Theorem 4.4.5. $\square$

Corollary 4.4.7. *Let f be an idempotent in a ring A that has a nilpotent ideal I. Then f is primitive if and only if $f + I$ is primitive.*

Proof. We have seen in Theorem 4.4.5 that if $f+I$ is primitive, then so is f. Conversely, if $f+I$ can be written $f+I = e_1 + e_2$ where the e_i are orthogonal idempotents of A/I, then by applying Corollary 4.4.6 to the ring fAf (of which f is the identity) we may write $f = g_1 + g_2$ where the g_i are orthogonal idempotents of A that lift the e_i. □

We now classify the indecomposable projective modules over a finite dimensional algebra as the projective covers of the simple modules. We first describe how these projective covers arise, and then show that they exhaust the possibilities for indecomposable projective modules. We postpone explicit examples until the next section, in which we consider group algebras.

Theorem 4.4.8. *Let A be a finite dimensional algebra over a field and S a simple A-module.*

(1) *There is an indecomposable projective module P_S with $P_S/\operatorname{Rad} P_S \cong S$, of the form $P_S = Af$ where f is a primitive idempotent in A.*

(2) *The idempotent f has the property that $fS \neq 0$ and if T is any simple module not isomorphic to S then $fT = 0$.*

(3) *P_S is the projective cover of S, it is uniquely determined up to isomorphism by this property and has S as its unique simple quotient.*

(4) *It is also possible to find an idempotent $f_S \in A$ so that $f_S S = S$ and $f_S T = 0$ for every simple module T not isomorphic to S.*

Proof. Let $e \in A/\operatorname{Rad} A$ be any primitive idempotent such that $eS \neq 0$. It is possible to find such e since we may write 1 as a sum of primitive idempotents and some term in the sum must be non-zero on S. Let f be any lift of e to A, possible by Corollary 4.4.7. Then f is primitive, $fS = eS \neq 0$ and $fT = eT = 0$ if $T \not\cong S$ since a primitive idempotent e in the semisimple ring $A/\operatorname{Rad} A$ is non-zero on a unique isomorphism class of simple modules. We define $P_S = Af$, an indecomposable projective module. Now

$$P_S/\operatorname{Rad} P_S = Af/(\operatorname{Rad} A \cdot Af) \cong (A/\operatorname{Rad} A) \cdot (f + \operatorname{Rad} A) = S,$$

the isomorphism arising because the map $Af \to (A/\operatorname{Rad} A) \cdot (f + \operatorname{Rad} A)$ defined by $af \mapsto (af + \operatorname{Rad} A)$ has kernel $(\operatorname{Rad} A) \cdot f$. The fact that P_S is the projective cover of S is a consequence of Nakayama's lemma, and the uniqueness of the projective cover was dealt with in Proposition 4.4.3. Any simple quotient of P_S is a quotient of $P_S/\operatorname{Rad} P_S$, so there is only one of these. Finally we observe that if we had written 1 as a sum of primitive central idempotents in $A/\operatorname{Rad} A$, the lift of the unique such idempotent that is non-zero on S is the desired idempotent f_S. □

Theorem 4.4.9. *Let A be a finite dimensional algebra over a field k. Up to isomorphism, the indecomposable projective A-modules are exactly the modules P_S that are the projective covers of the simple modules, and $P_S \cong P_T$ if and only if $S \cong T$. Each*

projective P_S appears as a direct summand of the regular representation, with multiplicity equal to the multiplicity of S as a summand of $A/\operatorname{Rad} A$. As a left A-module the regular representation decomposes as

$$A \cong \bigoplus_{\text{simple } S} (P_S)^{n_S}$$

where $n_S = \dim_k S$ if k is algebraically closed, and more generally $n_S = \dim_D S$ where $D = \operatorname{End}_A(S)$.

In what follows we will only prove that finitely generated indecomposable projective modules are isomorphic to P_S, for some simple S. In Exercise 4.8.10 at the end of this chapter it is shown that this accounts for all indecomposable projective modules.

Proof. Let P be an indecomposable projective module and write

$$P/\operatorname{Rad} P \cong S_1 \oplus \cdots \oplus S_n.$$

Then $P \to S_1 \oplus \cdots \oplus S_n$ is a projective cover. Now

$$P_{S_1} \oplus \cdots \oplus P_{S_n} \to S_1 \oplus \cdots \oplus S_n$$

is also a projective cover, and by uniqueness of projective covers we have

$$P \cong P_{S_1} \oplus \cdots \oplus P_{S_n}.$$

Since P is indecomposable we have $n = 1$ and $P \cong P_{S_1}$.

Suppose that each simple A module S occurs with multiplicity n_S as a summand of the semisimple ring $A/\operatorname{Rad} A$. Both A and $\bigoplus_{\text{simple } S} P_S^{n_S}$ are the projective cover of $A/\operatorname{Rad} A$, and so they are isomorphic. We have seen in Corollary 3.4.4 that $n_S = \dim_k S$ when k is algebraically closed, and in Exercise 3.6.5 of Chapter 2 that $n_S = \dim_D S$ in general. $\qquad\square$

Theorem 4.4.10. *Let A be a finite dimensional algebra over a field k, and U an A-module. Then U has a projective cover.*

Again, we only give a proof in the case that U is finitely generated, leaving the general case to Exercise 4.8.10 of this chapter.

Proof. Since $U/\operatorname{Rad} U$ is semisimple we may write $U/\operatorname{Rad} U = S_1 \oplus \cdots \oplus S_n$, where the S_i are simple modules. Let P_{S_i} be the projective cover of S_i and $h : P_{S_1} \oplus \cdots \oplus P_{S_n} \to U/\operatorname{Rad} U$ the projective cover of $U/\operatorname{Rad} U$. By projectivity there exists a homomorphism f such that the following diagram commutes:

$$
\begin{array}{ccc}
& P_{S_1} \oplus \cdots \oplus P_{S_n} & \\
{\scriptstyle f}\swarrow & & \downarrow{\scriptstyle h} \\
U & \xrightarrow{\ g\ } & U/\operatorname{Rad} U
\end{array}
\quad.
$$

Since both g and h are essential epimorphisms, so is f by Proposition 4.4.2. Therefore f is a projective cover. $\qquad\square$

We should really learn more from Theorem 4.4.10 than simply that U has a projective cover: the projective cover of U is the same as the projective cover of $U/\operatorname{Rad} U$.

Example 4.4.11. The arguments that show the existence of projective covers have a sense of inevitability about them and we may get the impression that projective covers always exist in arbitrary situations. In fact they fail to exist in general for integral group rings. If $G = \{e, g\}$ is a cyclic group of order 2, consider the submodule $3\mathbb{Z} \cdot e + \mathbb{Z} \cdot (e+g)$ of $\mathbb{Z}G$ generated as an abelian group by $3e$ and $e+g$. We rapidly check that this subgroup is invariant under the action of G (so it is a $\mathbb{Z}G$-submodule), and it is not the whole of $\mathbb{Z}G$ since it does not contain e. Applying the augmentation map $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ we have $\epsilon(3e) = 3$ and $\epsilon(e+g) = 2$ so $\epsilon(3\mathbb{Z} \cdot e + \mathbb{Z} \cdot (e+g)) = 3\mathbb{Z} + 2\mathbb{Z} = \mathbb{Z}$. This shows that the epimorphism ϵ is not essential, and so it is not a projective cover of $\mathbb{Z}$. If $\mathbb{Z}$ were to have a projective cover it would be a proper summand of $\mathbb{Z}G$ by Proposition 4.4.3. On reducing modulo 2 we would deduce that $\mathbb{F}_2 G$ decomposes, which we know not to be the case by Corollary 4.1.7. This shows that $\mathbb{Z}$ has no projective cover as a $\mathbb{Z}G$-module.

4.5 The Cartan matrix

Now that we have classified the projective modules for a finite dimensional algebra we turn to one of their important uses, which is to determine the multiplicity of a simple module S as a composition factor of an arbitrary module U (with a composition series). If

$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

is any composition series of U, the number of quotients U_i/U_{i-1} isomorphic to S is determined independently of the choice of composition series, by the Jordan–Hölder theorem. We call this number the *(composition factor) multiplicity* of S in U.

Proposition 4.5.1. *Let S be a simple module for a finite dimensional algebra A with projective cover P_S, and let U be a finite dimensional A-module.*

(1) *If T is a simple A-module then*

$$\dim \operatorname{Hom}_A(P_S, T) = \begin{cases} \dim \operatorname{End}_A(S) & \text{if } S \cong T, \\ 0 & \text{otherwise.} \end{cases}$$

(2) *The multiplicity of S as a composition factor of U is*

$$\dim \operatorname{Hom}_A(P_S, U) / \dim \operatorname{End}_A(S).$$

(3) *If $e \in A$ is an idempotent then $\dim \operatorname{Hom}_A(Ae, U) = \dim eU$.*

We remind the reader that if the ground field k is algebraically closed then $\dim \operatorname{End}_A(S) = 1$ by Schur's lemma. Thus the multiplicity of S in U is just $\dim \operatorname{Hom}_A(P_S, U)$ in this case.

Proof. (1) If $P_S \to T$ is any non-zero homomorphism, the kernel must contain $\operatorname{Rad} P_S$, being a maximal submodule of P_S. Since $P_S / \operatorname{Rad} P_S \cong S$ is simple, the kernel must be $\operatorname{Rad} P_S$ and $S \cong T$. Every homomorphism $P_S \to S$ is the composite $P_S \to P_S / \operatorname{Rad} P_S \to S$ of the quotient map and either an isomorphism of $P_S / \operatorname{Rad} P_S$ with S or the zero map. This gives an isomorphism $\operatorname{Hom}_A(P_S, S) \cong \operatorname{End}_A(S)$.

(2) Let
$$0 = U_0 \subset U_1 \subset \cdots \subset U_n = U$$

be a composition series of U. We prove the result by induction on the composition length n, the case $n = 1$ having just been established. Suppose $n > 1$ and that the multiplicity of S in U_{n-1} is $\dim \operatorname{Hom}_A(P_S, U_{n-1}) / \dim \operatorname{End}_A(S)$. The exact sequence

$$0 \to U_{n-1} \to U \to U/U_{n-1} \to 0$$

gives rise to an exact sequence

$$0 \to \operatorname{Hom}_A(P_S, U_{n-1}) \to \operatorname{Hom}_A(P_S, U) \to \operatorname{Hom}_A(P_S, U/U_{n-1}) \to 0$$

by Proposition 4.2.3, so that

$$\dim \operatorname{Hom}_A(P_S, U) = \dim \operatorname{Hom}_A(P_S, U_{n-1}) + \dim \operatorname{Hom}_A(P_S, U/U_{n-1}).$$

Dividing these dimensions by $\dim \operatorname{End}_A(S)$ gives the result, by part (1).

(3) There is an isomorphism of vector spaces $\operatorname{Hom}_A(Ae, U) \cong eU$ specified by $\phi \mapsto \phi(e)$. Note here that since $\phi(e) = \phi(ee) = e\phi(e)$ we must have $\phi(e) \in eU$. This mapping is injective since each A-module homomorphism $\phi : Ae \to U$ is determined by its value on e as $\phi(ae) = a\phi(e)$. It is surjective since the equation just written down does define a module homomorphism for each choice of $\phi(e) \in eU$. $\qquad \square$

Again in the context of a finite dimensional algebra A, we define for each pair of simple A-modules S and T the integer

$$c_{ST} = \text{the composition factor multiplicity of } S \text{ in } P_T.$$

These are called the *Cartan invariants* of A, and they form a matrix $C = (c_{ST})$ with rows and columns indexed by the isomorphism types of simple A-modules, called the *Cartan matrix* of A.

Corollary 4.5.2. *Let A be a finite dimensional algebra over a field, let S and T be simple A-modules and let e_S, e_T be idempotents so that $P_S = Ae_S$ and $P_T = Ae_T$ are projective covers of S and T. Then*

$$c_{ST} = \dim \operatorname{Hom}_A(P_S, P_T) / \dim \operatorname{End}_A(S) = \dim e_S A e_T / \dim \operatorname{End}_A(S).$$

If the ground field k is algebraically closed then $c_{ST} = \dim \operatorname{Hom}_A(P_S, P_T) = \dim e_S A e_T$.

While it is rather weak information just to know the composition factors of the projective modules, this is at least a start in describing these modules. We will see later on in the case of group algebras that there is an extremely effective way of computing the Cartan matrix using the decomposition matrix.

4.6 Dualities, injective modules and the Nakayama functor

Definition 4.6.1. A *duality* between two categories $\mathcal{C}, \mathcal{D}$ is a contravariant equivalence of categories $F : \mathcal{C}^{\mathrm{op}} \to \mathcal{D}$.

Usually this term is only used when the categories have some additive structure. We will describe two dualities that are available for representations of a finite dimensional algebra A over a field K. Notice that the category mod-A of right A-modules is equivalent to the category A^{op}-mod of left modules for the opposite ring A^{op}. If A is the path algebra of a quiver, then A^{op} is the path algebra of the opposite quiver.

Definition 4.6.2. The first duality we consider is the functor $D : A$-mod $\to$ mod-A between finite dimensional left and right A-modules, defined by $D(M) := \operatorname{Hom}_K(M, K)$, the vector space dual. If M is a left A-module then $D(M)$ becomes a right A-module using the left action of A on M.

This functor D has an inverse duality, also denoted D and defined by the same formula. The fact that the natural map $M \to DD(M)$ is an isomorphism shows that D is a duality. The duality D is exact: it sends a short exact sequence of left A-modules $0 \to L \to M \to N \to 0$ to a short exact sequence of right A-modules $0 \to DN \to DM \to Dl \to 0$. Thus, the lattice of submodules of DM is the opposite of the lattice of submodules of M. A module S is simple if and only if DS is simple. Furthermore, if S is associated to an idempotent $e \in A$, in the sense that $eS \neq 0$ but $eT = 0$ if $S \not\cong T$, then DS is also associated to e. Direct sum decompositions are preserved by D, so M is indecomposable if and only if DM is indecomposable, and M is semisimple if and only if DM is semisimple. The quotient map $M \to M/\operatorname{Rad}(M)$ is exchanged by D with the inclusion of the socle $\operatorname{Soc}(DM) \to DM$. Uniserial modules are sent to uniserial modules by D. Projective modules are sent to injective modules by D, and vice-versa. The Loewy lengths of M and DM are the same.

We may define an *essential monomorphism* $f : L \to M$ to be a monomorphism with the property that whenever $g : M \to N$ is a homomorphism for which gf is a monomorphism, then g is a monomorphism. The theory of essential monomorphisms proceeds in exactly the same way as the theory of essential epimorphisms, with a two out of three lemma, and so on. A dual version of Nakayama's lemma (for modules over a finite dimensional algebra) states that the inclusion $\operatorname{Soc}(M) \to M$ is an essential monomorphism We see that D exchanges essential epimorphisms with essential monomorphisms.

Definition 4.6.3. Given a module M, an essential monomorphism $M \to I$, where I is an injective module, is called an *injective envelope* or *injective hull* of M.

The injective envelope of M is unique, if it exists.

Proposition 4.6.4. *Let A be a finite dimensional algebra over a field K. The indecomposable injective A-modules are precisely the injective envelopes I_S of the simple*

A-modules *S* and $\mathrm{Soc}(I_S)$ *is the unique simple submodule of* I_S. *Every module has an injective envelope.*

Example 4.6.5. We describe the indecomposable injective modules for the quivers $Q_1 = 1 \to 2 \to 3$ and $Q_2 = 1 \to 2 \leftarrow 3$.

We now consider the contravariant functor $(-)^\vee : A\text{-mod} \to \text{mod-}A$ defined on each left *A*-module *M* as $M^\vee := \mathrm{Hom}_A(M, A)$. This time, if *M* is a left *A*-module then $M^\vee$ becomes a right *A*-module using the right action on the second variable *A*. We also get a functor in the reverse direction defined by the same formula, whose effect on a right module *M* is also denoted $M^\vee$. This functor is left exact, but not, in general, exact. It will only be exact on all modules *M* if *A* is an injective module (as well as being projective). Thus $(-)^\vee$ is not a duality on the whole module category.

Proposition 4.6.6. *Let A be a ring with a 1. The functor* $(-)^\vee$ *restricts to a contravariant equivalence between the full subcategories of A-mod and of mod-A whose objects are the projective modules. If A is a finite dimensional algebra over a field K, an indecomposable projective Af corresponding to an idempotent f has* $(Af)^\vee \cong fA$.

Proof. Taking $M = {}_A A$ we have that $A^\vee = \mathrm{Hom}_A(A, A) \cong A_A$ is projective and that the natural map $A \to A^{\vee\vee}$ is an isomorphism. It follows that the same is true when *M* is projective, because such a module is a summand of a direct sum of copies of ${}_A A$. This demonstrates the claimed equivalence of categories. When *A* is a finite dimensional algebra, $(Af)^\vee = \mathrm{Hom}_A(Af, A) \cong fA$ via an isomorphism that sends a homomorphism $\phi : Af \to A$ to $\phi(f)$. Note that, since $\phi(f) = \phi(f^2) = f\phi(f)$, necessarily $\phi(f) \in fA$. This homomorphism has inverse that sends an element fa to the homomorphism $\phi(bf) = bfa$. $\square$

When *A* is a finite dimensional algebra we define the Nakayama functor $\nu : A\text{-mod} \to A\text{-mod}$ to be the composite of the two contravariant functors we have just considered. Thus if *M* is a left *A*-module then $\nu(M) := D(M^\vee)$.

Proposition 4.6.7. *Let A be a finite dimensional algebra over a field K.*

1. *The Nakayama functor* ν *is right exact.*

2. ν *provides an equivalence of categories between the full subcategory of A-mod whose objects are projective and the full subcategory of A-mod whose objects are injective, with inverse functor that sends N to* $(DN)^\vee$. *We have* $\nu(P_S) \cong I_S$ *for each simple module S.*

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Proof. The right exactness comes because *D* is exact and $(-)^\vee$ is left exact. Both of these functors are their own inverse on the subcategories in question, and the formula for the inverse equivalence follows from this. The fact that $\nu(P_S) \cong I_S$ comes from the fact that both contravariant functors *D* and $(-)^\vee$ preserve the correspondence of simples, indecomposable projectives and indecomposable injectives with idempotents. $\square$

Example 4.6.8. The endomorphism ring of the direct sum of the indecomposable projectives is isomorphic to the endomorphism ring of the direct sum of the indecomposable injectives. We illustrate this with Q_1 and Q_2 considered earlier.

4.7 Summary of Chapter 4

- Direct sum decompositions of $_AA$ as an A-module (with indecomposable summands) correspond to expressions for 1_A as a sum of orthogonal (primitive) idempotents.

- $U \to U/\operatorname{Rad} U$ is essential.

- Projective covers are unique when they exist. For modules for a finite dimensional algebra over a field they do exist.

- Idempotents can be lifted through nilpotent ideals.

- The indecomposable projective modules for a finite dimensional algebra over a field are exactly the projective covers of the simple modules. Each has a unique simple quotient and is a direct summand of the regular representation. Over an algebraically closed field P_S occurs as a summand of the regular representation with multiplicity $\dim S$.

4.8 Exercises for Chapter 4

Exercise 4.8.1. Let A be a finite dimensional algebra over a field. Show that A is semisimple if and only if all finite dimensional A-modules are projective.

Exercise 4.8.2. Let P_S be an indecomposable projective module for a finite dimensional algebra over a field. Show that every non-zero homomorphic image of P_S
    (a) has a unique maximal submodule,
    (b) is indecomposable, and
    (c) has P_S as its projective cover.

Exercise 4.8.3. Let A be a finite dimensional algebra over a field, and suppose that f, f' are primitive idempotents of A. Show that the indecomposable projective modules Af and Af' are isomorphic if and only if $\dim fS = \dim f'S$ for every simple module S.

Exercise 4.8.4. Let A be a finite dimensional algebra over a field and $f \in A$ a primitive idempotent. Show that there is a simple A-module S with $fS \neq 0$, and that S is uniquely determined up to isomorphism by this property.

Exercise 4.8.5. Let A be a finite dimensional algebra over a field, and suppose that Q is a projective A-module. Show that in any expression

$$Q = P_{S_1}^{n_1} \oplus \cdots \oplus P_{S_r}^{n_r}$$

where $S_1, \ldots, S_r$ are non-isomorphic simple modules, we have

$$n_i = \dim \operatorname{Hom}_A(Q, S_i) / \dim \operatorname{End}_A(S_i).$$

Exercise 4.8.6. Let A be a finite dimensional algebra over a field. Suppose that V is an A-module, and that a certain simple A-module S occurs as a composition factor of V with multiplicity 1. Suppose that there exist non-zero homomorphisms $S \to V$ and $V \to S$. Prove that S is a direct summand of V.

Exercise 4.8.7. Let $G = S_n$, let k be a field of characteristic 2 and let $\Omega = \{1, 2, \ldots, n\}$ permuted transitively by G.
   (a) When $n = 3$, show that the permutation module $k\Omega$ is semisimple, being the direct sum of the one-dimensional trivial module and the 2-dimensional simple module. [Use the information from Example 4.3.2 and Exercise **??** from Chapter 6.]
   (b) When $n = 4$ there is a normal subgroup $V \lhd S_4$ with $S_4/V \cong S_3$, where $V = \langle (1,2)(3,4), (1,3)(2,4) \rangle$. Show that the simple kS_4-modules are precisely the two simple kS_3-modules, made into kS_4-modules via the quotient homomorphism to S_3. Show that $k\Omega$ is uniserial with three composition factors that are the trivial module, the 2-dimensional simple module and the trivial module.
[Use Exercise **??** from Chapter 6.]

Exercise 4.8.8. Show by example that if H is a subgroup of G it need not be true that $\operatorname{Rad} kH \subseteq \operatorname{Rad} kG$.
[Compare this result with Exercise **??** from Chapter 6.]

Exercise 4.8.9. Suppose that we have module homomorphisms $U \xrightarrow{f} V \xrightarrow{g} W$. Show that part of Proposition 4.4.2(1) can be strengthened to say the following: if gf is an essential epimorphism and f is an epimorphism then both f and g are essential epimorphisms.

Exercise 4.8.10. Let U and V be arbitrary (not necessarily Noetherian) modules for a finite dimensional algebra A. Use the results of Exercise **??** of Chapter 6 to show the following.
   (a) Show that the quotient homomorphism $U \to U/\operatorname{Rad} U$ is essential.
   (b) Show that a homomorphism $U \to V$ is essential if and only if the homomorphism of radical quotients $U/\operatorname{Rad} U \to V/\operatorname{Rad} V$ is an isomorphism.
   (c) Show that U has a projective cover.
   (d) Show that every indecomposable projective A-module is finite dimensional, and hence isomorphic to P_S for some simple module S.
   (e) Show that every projective A-module is a direct sum of indecomposable projective modules.

Exercise 4.8.11. In this question U, V and W are modules for a finite dimensional algebra over a field and P_W is the projective cover of W. Assume either that these modules are finite dimensional, or the results from the last exercise.

(a) Show that $U \to W$ is an essential epimorphism if and only if there is a surjective homomorphism $P_W \to U$ so that the composite $P_W \to U \to W$ is a projective cover of W. In this situation show that $P_W \to U$ must be a projective cover of U.

(b) Prove the following 'extension and converse' to Nakayama's lemma: let V be any submodule of U. Then $U \to U/V$ is an essential epimorphism $\Leftrightarrow V \subseteq \operatorname{Rad} U$.

Chapter 5

Indecomposable modules and Auslander-Reiten theory

5.1 The endomorphism ring of an indecomposable module

Proposition 5.1.1. *Let U be a module for a ring A with a 1. Expressions*

$$U = U_1 \oplus \cdots \oplus U_n$$

as a direct sum of submodules biject with expressions $1_U = e_1 + \cdots + e_n$ for the identity $1_U \in \operatorname{End}_A(U)$ as a sum of orthogonal idempotents. Here e_i is obtained from U_i as the composite of projection and inclusion $U \to U_i \to U$, and U_i is obtained from e_i as $U_i = e_i(U)$. The summand U_i is indecomposable if and only if e_i is primitive.

Proof. We must check several things. Two constructions are indicated in the statement of the proposition: given a direct sum decomposition of U we obtain an idempotent decomposition of 1_U, and vice-versa. It is clear that the idempotents constructed from a module decomposition are orthogonal and sum to 1_U. Conversely, given an expression $1_U = e_1 + \cdots + e_n$ as a sum of orthogonal idempotents, every element $u \in U$ can be written $u = e_1 u + \cdots + e_n u$ where $e_i u \in e_i U = U_i$. In any expression $u = u_1 + \cdots u_n$ with $u_i \in e_i U$ we have $e_j u_i \in e_j e_i U = 0$ if $i \neq j$ so $e_i u = e_i u_i = u_i$, and this expression is uniquely determined. Thus the expression $1_U = e_1 + \cdots + e_n$ gives rise to a direct sum decomposition.

We see that U_i decomposes as $U_i = V \oplus W$ if and only if $e_i = e_V + e_W$ can be written as a sum of orthogonal idempotents, and so U_i is indecomposable if and only if e_i is primitive. $\square$

Corollary 5.1.2. *An A-module U is indecomposable if and only if the only non-zero idempotent in $\operatorname{End}_A(U)$ is 1_U.*

Proof. From the proposition, U is indecomposable if and only if 1_U is primitive, and this happens if and only if 1_U and 0 are the only idempotents in $\operatorname{End}_A(U)$. This last implication in the forward direction follows since any idempotent e gives rise to

an expression $1_U = e + (1_U - e)$ as a sum of orthogonal idempotents, and in the opposite direction there simply are no non-trivial idempotents to allow us to write $1_U = e_1 + e_2$. $\square$

The equivalent conditions of the next result are satisfied by the endomorphism ring of an indecomposable module, but we first present them in abstract. The connection with indecomposable modules will be presented in Corollary 5.1.5.

Proposition 5.1.3. *Let B be a ring with 1. The following are equivalent.*

(1) B has a unique maximal left ideal.

(2) B has a unique maximal right ideal.

(3) $B/\operatorname{Rad}(B)$ is a division ring.

(4) The set of elements in B that are not invertible forms a left ideal.

(5) The set of elements in B that are not invertible forms a right ideal.

(6) The set of elements in B that are not invertible forms a 2-sided ideal.

Proof. (1) $\Rightarrow$ (3) Let I be the unique maximal left ideal of B. Since $\operatorname{Rad}(B)$ is the intersection of the maximal left ideals, it follows that $I = \operatorname{Rad}(B)$. If $a \in B - I$ then Ba is a left ideal not contained in I, so $Ba = B$. Thus there exists $x \in B$ with $xa = 1$. Furthermore $x \notin I$, so $Bx = B$ also and there exists $y \in B$ with $yx = 1$. Now $yxa = a = y$ so a and x are 2-sided inverses of one another. This implies that B/I is a division ring.

(1) $\Rightarrow$ (6) The argument just presented shows that the unique maximal left ideal I is in fact a 2-sided ideal, and every element not in I is invertible. This implies that every non-invertible element is contained in I. Equally, no element of I can be invertible, so I consists of the non-invertible elements, and they form a 2-sided ideal.

(3) $\Rightarrow$ (1) If I is a maximal left ideal of B then $I \supseteq \operatorname{Rad}(B)$ and so corresponds to a left ideal of $B/\operatorname{Rad}(B)$, which is a division ring. It follows that either $I = \operatorname{Rad}(B)$ or $I = B$, and so $\operatorname{Rad}(B)$ is the unique maximal left ideal of B.

(4) $\Rightarrow$ (1) Let J be the set of non-invertible elements of B and I a maximal left ideal. Then no element of I is invertible, so $I \subseteq J$. Since J is an ideal, we have equality, and I is unique.

(6) $\Rightarrow$ (4) This implication is immediate, and so we have established the equivalence of conditions (1), (3), (4) and (6).

Since conditions (3) and (6) are left-right symmetric, it follows that they are also equivalent to conditions (2) and (5), by analogy with the equivalence with (1) and (4). $\square$

We will call a ring B satisfying any of the equivalent conditions of the last proposition a *local ring*. Any commutative ring that is local in the usual sense (i.e. it has

a unique maximal ideal) is evidently local in this non-commutative sense. As for non-commutative examples of local rings, we see from Proposition 4.1.3 part (3) that if G is a p-group and k is a field of characteristic p then the group algebra kG is a local ring. This is because its radical is the augmentation ideal and the quotient by the radical is k, which is a division ring, thus verifying condition (3) of Proposition 5.1.3.

We have seen in Corollary 5.1.2 a characterization of indecomposable modules as modules whose endomorphism ring only has idempotents 0 and 1. We now make the connection with local rings.

Proposition 5.1.4. *(1) In a local ring the only idempotents are 0 and 1.*

(2) Suppose that B is an R-algebra that is finitely generated as an R-module, where R is a complete discrete valuation ring or a field. If the only idempotents in B are 0 and 1 then B is a local ring.

Proof. (1) In a local ring B, any idempotent e other than 0 and 1 would give a non-trivial direct sum decomposition of $B = Be \oplus B(1 - e)$ as left B-modules, and so B would have more than one maximal left ideal, a contradiction.

(2) Suppose that 0 and 1 are the only idempotents in B, and let (π) be the maximal ideal of R. Just as in the proof of part (1) of Proposition **??** we see that π annihilates every simple B-module, and so $\pi B \subseteq \operatorname{Rad}(B)$. This implies that $B/\operatorname{Rad}(B)$ is a finite dimensional $R/(\pi)$-algebra. If $e \in B/\operatorname{Rad}(B)$ is idempotent then by the argument of Proposition **??** it lifts to an idempotent of B, which must be 0 or 1. Since e is the image of this lifting, it must also be 0 or 1. Now $B/\operatorname{Rad}(B) \cong M_{n_1}(\Delta_1) \oplus \cdots \oplus M_{n_t}(\Delta_t)$ for certain division rings Δ_i, since this is a semisimple algebra, and the only way this algebra would have just one non-zero idempotent is if $t = 1$ and $n_1 = 1$. This shows that condition (3) of the last proposition is satisfied. $\qquad\square$

We put these pieces together:

Corollary 5.1.5. *Let U be a module for a ring A.*

(1) If $\operatorname{End}_A(U)$ is a local ring then U is indecomposable.

(2) Suppose that R is a complete discrete valuation ring or a field, A is an R-algebra, and U is finitely-generated as an R-module. Then U is indecomposable if and only if $\operatorname{End}_A(U)$ is a local ring. In particular this holds if $A = RG$ where G is a finite group.

Proof. (1) This follows from Corollary 5.1.2 and Proposition 5.1.4.

(2) From Corollary 5.1.2 and Proposition 5.1.4 again all we need to do is to show that $\operatorname{End}_A(U)$ is finitely-generated as an R-module. Let $R^m \to U$ be a surjection of R-modules. Composition with this surjection gives a homomorphism $\operatorname{End}_A(U) \to \operatorname{Hom}_R(R^m, U)$, and it is an injection since $R^m \to U$ is surjective (using the property of Hom from homological algebra that it is 'left exact' and the fact that A-module homomorphisms are a subset of R-module homomorphisms). Thus $\operatorname{End}_A(U)$ is realized

as an R-submodule of $\operatorname{Hom}_R(R^m, U) \cong U^m$, which is a finitely generated R-module. Since R is Noetherian, the submodule is also finitely-generated. □

The next result is a version of the Krull-Schmidt theorem. We first present it in greater generality than for group representations.

Theorem 5.1.6 (Krull-Schmidt). *Let A be a ring with a 1, and suppose that U is an A-module that has two A-module decompositions*

$$U = U_1 \oplus \cdots \oplus U_r = V_1 \oplus \cdots \oplus V_s$$

where, for each i, $\operatorname{End}_A(U_i)$ is a local ring and V_i is an indecomposable A-module. Then $r = s$ and the summands U_i and V_j are isomorphic in pairs when taken in a suitable order.

Proof. The proof is by induction on $\max\{r, s\}$. When this number is 1 we have $U = U_1 = V_1$, and this starts the induction.

Now suppose $\max\{r, s\} > 1$ and the result is true for smaller values of $\max\{r, s\}$. For each j let $\pi_j : U \to V_j$ be projection onto the jth summand with respect to the decomposition $U = V_1 \oplus \cdots \oplus V_s$, and let $\iota_j : V_j \hookrightarrow U$ be inclusion. Then $\sum_{j=1}^s \iota_j \pi_j = 1_U$. Now let $\beta : U \to U_1$ be projection with respect to the decomposition $U = U_1 \oplus \cdots \oplus U_r$ and $\alpha : U_1 \hookrightarrow U$ be inclusion so that $\beta\alpha = 1_{U_1}$. We have

$$1_{U_1} = \beta \left(\sum_{j=1}^s \iota_j \pi_j \right) \alpha = \sum_{j=1}^s \beta \iota_j \pi_j \alpha$$

and since $\operatorname{End}_A(U_1)$ is a local ring it follows that at least one term $\beta \iota_j \pi_j \alpha$ must be invertible. By renumbering the V_j if necessary we may suppose that $j = 1$, and we write $\phi = \beta \iota_1 \pi_1 \alpha$. Now $(\phi^{-1}\beta\iota_1)(\pi_1\alpha) = 1_{U_1}$ and so $\pi_1\alpha : U_1 \to V_1$ is split mono and $\phi^{-1}\beta\iota_1 : V_1 \to U_1$ is split epi. It follows that $\pi_1\alpha(U_1)$ is a direct summand of V_1. Since V_1 is indecomposable we have $\pi_1\alpha(U_1) = V_1$ and $\pi_1\alpha : U_1 \to V_1$ must be an isomorphism.

We now show that $U = U_1 \oplus V_2 \oplus \cdots \oplus V_s$. Because $\pi_1\alpha$ is an isomorphism, π_1 is one-to-one on the elements of U_1. Also π_1 is zero on $V_2 \oplus \cdots \oplus V_s$ and it follows that $U_1 \cap (V_2 \oplus \cdots \oplus V_s) = 0$, since any element of the intersection is detected by its image under π_1, and this must be zero. The submodule $U_1 + V_2 + \cdots + V_s$ contains $V_2 + \cdots + V_s = \ker \pi_1$ and so corresponds via the first isomorphism theorem for modules to a submodule of $\pi_1(U) = V_1$. In fact π_1 is surjective and so $U_1 + V_2 + \cdots + V_s = U$. It follows that $U = U_1 \oplus V_2 \oplus \cdots \oplus V_s$.

We now deduce that $U/U_1 \cong U_2 \oplus \cdots \oplus U_r \cong V_2 \oplus \cdots \oplus V_s$. It follows by induction that $r = s$ and the summands are isomorphic in pairs, which completes the proof. □

Note that the proof of Theorem 5.1.6 shows that an 'exchange lemma' property holds for the indecomposable summands in the situation of the theorem. After the abstraction of general rings, we state the Krull-Schmidt theorem in the context of finite group representations, just to make things clear.

Corollary 5.1.7. *Let R be a complete discrete valuation ring or a field and G a finite group. Suppose that U is a finitely-generated RG-module that has two decompositions*

$$U = U_1 \oplus \cdots \oplus U_r = V_1 \oplus \cdots \oplus V_s$$

where the U_i and V_j are indecomposable RG-modules. Then $r = s$ and the summands U_i and V_j are isomorphic in pairs when taken in a suitable order.

Proof. We have seen in Corollary 5.1.5 that the rings $\mathrm{End}_{RG}(U_i)$ are local, so that Theorem 5.1.6 applies. $\square$

5.2 Irreducible morphisms

An element a of a commutative ring is irreducible if it is not a unit, and whenever $a = bc$ then b is a unit or c is a unit. In the case of the ring $\mathbb{Z}$ this is the definition that most people would give of a prime number, although in mathematics we use a different definition for this. Such irreducible elements have importance when it comes to questions of unique factorization. It is not completely obvious at the beginning how to generalize this definition to non-commutative situations, and whether such a generalization would be useful. We now give the definition in the context of categories, and it turns out to be extremely useful.

Definition 5.2.1. A morphism $g : x \to y$ in a category $\mathcal{C}$ is called *irreducible* if g has neither a left inverse nor a right inverse, and whenever $g = ts$ for some morphisms $s : x \to z$ and $t : z \to y$ then either s has a left inverse or t has a right inverse.

This definition is most often applied in categories that have an additive structure, and especially in categories of modules and in triangulated categories. In this situation a morphism g has a left inverse if and only if it is split mono, and it has a right inverse if and only if it is split epi. In some texts a morphism that is split mono is called a *section* and a split epi is called a *retraction*. Thus the definition of irreducible morphism can be rephrased as follows:

Definition 5.2.2. A morphism $g : U \to U$ in the category of modules for some ring is called *irreducible* if g is neither split mono nor split epi, and whenever $g = ts$ then either s is split mono or t is split epi.

To understand this definition it helps to observe that, given any module homomorphism $g : U \to V$ and a further module W, we can always factorize g as

$$U \xrightarrow{\begin{bmatrix}1\\0\end{bmatrix}} U \oplus W \xrightarrow{\begin{bmatrix}g & 0\end{bmatrix}} V$$

and as

$$U \xrightarrow{\begin{bmatrix}g\\0\end{bmatrix}} V \oplus W \xrightarrow{\begin{bmatrix}1 & 0\end{bmatrix}} V.$$

These trivial factorizations are excluded as factorizations of an irreducible morphism because either the first morphism of the first factorization is split mono and the second morphism of the second factorization is split epi.

Here are some initial properties of irreducible morphisms.

Proposition 5.2.3. *Let $g : U \to V$ be an irreducible morphism in A-mod.*

1. *Either g is a monomorphism or an epimorphism.*

2. *Irreducible morphisms are preserved under equivalences of categories and dualities.*

3. *If g is a monomorphism then U is a summand of all proper submodules of V containing $g(U)$.*

4. *If g is an epimorphism then V is a summand of U/W for all submodules W of V such that $0 \neq W \subset \operatorname{Ker} g$.*

Proof. 1. We factor g as $U \xrightarrow{s} g(U) \xrightarrow{t} V$ and either s is (split) mono, in which case g is mono, or t is (split) epi, in which case g is epi.

2. follows because the definition of an irreducible morphism is in terms of categorical properties that are preserved under equivalence, and reversed under duality. The definition is self-dual.

3. Suppose that $g(U) \subseteq W \subset V$ with $W \neq V$, so that g factors as $U \xrightarrow{s} W \xrightarrow{t} V$. Here t is not epi, so that s is split mono.

4. This follows by a dual argument to the one that proved 3. □

Proposition 5.2.4. *Let A be a finite dimensional algebra.*

1. *Let P_S be a non-simple indecomposable projective A-module. The inclusion $\operatorname{Rad} P_S \to P_S$ is irreducible.*

2. *Let I_S be an indecomposable non-simple injective A-module. The quotient map $I_S \to I_S/\operatorname{Soc} I_S$ is irreducible.*

More is true than this: the inclusions of the summands of $\operatorname{Rad} P_S$ in P_S are irreducible, as are the surjections of I_S onto $I_S/\operatorname{Soc} I_S$.

Proof. 1. Consider the inclusion $f : \operatorname{Rad} P_S \to P_S$. This morphism is mono, but it is not split mono, because then we would have $P_S = \operatorname{Rad} P_S \oplus U$ for some submodule U so that $P_S = U$ by Nakayama's lemma and $\operatorname{Rad} P_S = 0$. Neither is this map split epi, because it is not epi. Consider a factorization $f = ts$ where $\operatorname{Rad} P_S \xrightarrow{s} W \xrightarrow{t} P_S$ for some module W. We will verify that either s is split mono or t is split epi. If t is epi then it is split epi because P_S is projective, so we may assume t is not epi. Now the image of t is contained in the unique maximal submodule $\operatorname{Rad} P_S$ of P_S, and it equals $\operatorname{Rad} P_S$ because this is the image of ts. We see from this that t provides a map $W \to \operatorname{Rad} P_S$ that is a left inverse to s, so that s is split mono.

2. This follows by duality from 1. □

Example 5.2.5. We consider the irreducible morphisms for the quiver $1 \to 2 \to 3$ that arise in this way.

Example 5.2.6. When $A = K[x]/(x^n)$ there is an indecomposable module $V_i = K[x]/(x^i)$ of each dimension i with $1 \le i \le n$. There are inclusion maps $V_i \to V_{i+1}$ and the quotient maps $V_{i+1} \to V_i$ and (up to isomorphism given by a commutative square) these are the irreducible maps between these modules.

5.3 The radical and the Auslander-Reiten quiver

Many (or perhaps all) of the definitions and results that follow apply in any *Krull-Schmidt category*, namely, an additive category where every object is isomorphic to a finite direct sum of indecomposable objects, and where the endomorphism rings of indecomposable objects are local rings. At times, what is written here does not make sense without assuming that homomorphism spaces are vector spaces over a field K.

Definition 5.3.1. Let A be a finite dimensional algebra and let U and V be finite dimensional A-modules. We define the *radical* of $\operatorname{Hom}_A(U, V)$ by

$$\operatorname{Rad}_A(U, V) = \{f : U \to V \mid hfg \text{ is never an isomorphism when } W \xrightarrow{g} U \xrightarrow{f} V \xrightarrow{h} W$$
$$\text{with } W \text{ indecomposable}\}$$

This is the form of the definition given in the book by Auslander, Reiten and Smaló, but it appears to be equivalent to

$$\operatorname{Rad}_A(U, V) = \{f : U \to V \mid \text{for all non-zero modules } W$$
$$\text{and morphisms } g : W \to U,\ h : V \to W,$$
$$hfg \text{ is never an isomorphism}\}.$$

To see that these definitions define the same set of morphisms, if f satisfies the condition of the second definition then it satisfies the condition of the first, because in the first we only have to test f with indecomposable modules W, rather than with all modules. Conversely, suppose that $f : U \to V$ satisfies the condition of the first definition, and consider morphisms $g : W \to U$ and $h : V \to W$ for some module W so that hfg is an isomorphism. If $W_1 \subseteq W$ is any indecomposable summand of W then $hfg(W_1)$ is also a summand of W, isomorphic to W_1 via an isomorphism $\alpha : hfg(W_1) \to W_1$. Letting $i : W_1 \to W$ be inclusion and $p : W \to hfg(W_1)$ be projection, the map $\alpha phfgi : W_1 \to W_1$ is an isomorphism. Because f satisfies the condition of definition 1 this cannot happen, so f satisfies the condition of definition 2.

Proposition 5.3.2. *Let A be a finite dimensional algebra over K and let U and V be finite dimensional A-modules. Then:*

1. $\operatorname{Rad}_A(U, V)$ *is a vector subspace of* $\operatorname{Hom}_A(U, V)$.

2. *If $L \xrightarrow{s} U \xrightarrow{f} V \xrightarrow{t} M$ are morphisms, where L, M are A-modules and $f \in$
   $\mathrm{Rad}_A(U, V)$ then $tfs \in \mathrm{Rad}_A(L, M)$.*

3. *If U and V are indecomposable A-modules then $\mathrm{Rad}_A(U, V)$ is the set of non-
   isomorphisms $U \to V$.*

4. *If $U = U_1 \oplus \cdots U_m$ and $V = V_1 \oplus \cdots \oplus V_n$ and homomorphisms are written as
   matrices $\phi = (\phi_{ij})$ with respect to this decomposition, where $\phi_{ij} : U_j \to V_i$, then
   $\phi \in \mathrm{Rad}_A(U, V)$ if and only if $\phi_{ij} \in \mathrm{Rad}_A(U_j, V_i)$.*

Proof. 1. We see that $\mathrm{Rad}_A(U, V)$ is closed under taking scalar multiples. Suppose both
f_1 and f_2 lie in $\mathrm{Rad}_A(U, V)$. If $g : W \to U$ and $h : V \to W$ with W indecomposable are
such that $h(f_1 + f_2)g = hf_1g + hf_2g$ is an isomorphism, then one of hf_1g and hf_2g must
be an isomorphism because $\mathrm{End}_A(W)$ is a local ring. This does not happen because f_1
and f_2 lie in the radical. Thus $f_1 + f_2$ lies in $\mathrm{Rad}_A(U, V)$.

2. If $g : W \to L$ and $h : M \to W$ are morphisms with W indecomposable it cannot
happen that $(ht)f(sg)$ is an isomorphism because f lies in the radical, so this shows
that tfs lies in the radical.

3. Suppose that U and V are indecomposable and that $f : U \to V$ is not an iso-
morphism. If $g : W \to U$ and $h : V \to W$ are morphisms with W indecomposable so
that hfg is an isomorphism then g must be split mono and h must be split epi. This
means that W is isomorphic to a direct summand of U, hence that g is an isomorphism
because U is indecomposable. Similarly h is an isomorphism, so also f is an isomor-
phism. This is not the case, so that f lies in $\mathrm{Rad}_A(U, V)$. Because $\mathrm{Rad}_A(U, V)$ does
not contain any isomorphism, it equals the set of non-isomorphisms.

4. Suppose that $\phi \in \mathrm{Rad}_A(U, V)$. We show that $\phi_{kj} \in \mathrm{Rad}_A(U_j, V_k)$. Now $\phi_{kj} =$
$p_k \phi i_j$ where p_k and i_j are projection and inclusion maps. For morphisms $g : W \to U_j$
and $h : V_k \to W$ we have $h\phi_{kj}g = hp_k\phi i_j g$ and this is not an isomorphism because
ϕ is in the radical. Conversely, suppose that $\phi_{kj} \in \mathrm{Rad}_A(U_j, V_k)$ for all k, j, and let
$g : W \to U$ and $h : V \to W$ with W indecomposable. Then

$$h\phi g = h\left(\sum_{k,j} i_k \phi_{kj} p_j\right)g = \sum_{k,j} h i_k \phi_{kj} p_j g$$

If this is an isomorphism then some term $h i_k \phi_{kj} p_j g$ in the sum must be an isomorphism,
because $\mathrm{End}_A(W)$ is a local ring. This cannot happen, because $\phi_{kj} \in \mathrm{Rad}_A(U_j, V_k)$.
Thus $\phi \in \mathrm{Rad}_A(U, V)$. $\qquad\square$

Condition 2. implies that in each variable, $\mathrm{Rad}_A(U, V)$ is a subfunctor of $\mathrm{Hom}_A(U, V)$.
Conditions 1. and 2. say that $\mathrm{Rad}_A(-, -)$ is a *2-sided ideal* in A-mod, in the terminol-
ogy used by Assem-Simson-Skowronski, and a *K-relation* on A-mod in the terminology
of Auslander-Reiten-Smalø. Condition 2. has the effect that we can form quotient func-
tors.

Corollary 5.3.3. *There are quotient functors $S_U : A$-mod $\to K$-mod and $S^V :$
$(A$-mod$)^{\mathrm{op}} \to k$-mod defined by*

$$S_U(V) = (\mathrm{Hom}_A(U, -)/\mathrm{Rad}_A(U, -))(V) = \mathrm{Hom}_A(U, V)/\mathrm{Rad}_A(U, V)$$

and

$$S^V(U) = (\operatorname{Hom}_A(-,V)/\operatorname{Rad}_A(-,V))(U) := \operatorname{Hom}_A(U,V)/\operatorname{Rad}_A(U,V).$$

These functors have the property that if M, U, V are indecomposable modules then

$$S_U(M) \cong \begin{cases} \operatorname{End}_A(U)/\operatorname{Rad}_A(U) & \text{if } M \cong U, \\ 0 & \text{otherwise.} \end{cases}$$

and

$$S^U(M) \cong \begin{cases} \operatorname{End}_A(U)/\operatorname{Rad}_A(U) & \text{if } M \cong U, \\ 0 & \text{otherwise.} \end{cases}$$

The non-zero quotients of these endomorphism rings are division rings. These functors are simple objects in the categories of K-linear functors A-mod $\to K$-mod and $(A$-mod$)^{\mathrm{op}} \to K$-mod.

When considering representations of A-mod and $(A$-mod$)^{\mathrm{op}}$ we only consider K-linear functors, that preserve the vector space structure of spaces of homomorphisms. There is much more to be said about this, but we postpone it.

Definition 5.3.4. We define the powers of the radical inductively by

$$\operatorname{Rad}_A^n(U,V) = \{f : U \to V \mid \text{there is a module } W$$
$$\text{and morphisms } g \in \operatorname{Rad}_A(U,W), \ h \in \operatorname{Rad}_A^{n-1}(W,V)$$
$$\text{with } f = hg\}.$$

We define the infinite radical by

$$\operatorname{Rad}_A^\infty(U,V) := \bigcap_{n>0} \operatorname{Rad}_A^n(U,V).$$

We might have expected to take sums of morphisms of the form hg in the above definition.

Lemma 5.3.5. $\operatorname{Rad}_A^n(U,V)$ *is a vector subspace of* $\operatorname{Hom}_A(U,V)$, *as is* $\operatorname{Rad}_A^\infty(U,V)$.

Proof. Suppose $f_i = h_i g_i$ with $i = 1, 2$ and $g_i \in \operatorname{Rad}_A(U,W_i)$ and $h_i \in \operatorname{Rad}_A^{n-1}(W_i,V)$. Then the linear combination $\lambda_1 f_1 + \lambda_2 f_2$ factors as

$$U \xrightarrow{\begin{bmatrix} g_1 \\ g_2 \end{bmatrix}} W_1 \oplus W_2 \xrightarrow{\begin{bmatrix} \lambda_1 h_1 & \lambda_2 h_2 \end{bmatrix}} V,$$

and the result is proved by induction on n. The intersection of vector subspaces is a vector subspace, which shows that $\operatorname{Rad}_A^\infty(U,V)$ is such. $\qquad\square$

Proposition 5.3.6. *Let $f : U \to V$ be a homomorphism of A-modules.*

1. *If U is indecomposable then $f \in \mathrm{Rad}_A(U, V)$ if and only if f is not split mono.*

2. *If V is indecomposable then $f \in \mathrm{Rad}_A(U, V)$ if and only if f is not split epi.*

3. *If both U and V are indecomposable then f is irreducible if and only if $f \in \mathrm{Rad}_A(U, V) - \mathrm{Rad}_A^2(U, V)$.*

Proof. 1. The morphism f is split mono if and only if there is a morphism $s : V \to U$ so that the composite $U \xrightarrow{f} V \xrightarrow{s} U = 1_U$, which is an isomorphism. This implies that f is not in the radical, from the definition. Conversely, if f is not in the radical there exists a module W and morphisms $W \xrightarrow{g} U \xrightarrow{f} V \xrightarrow{h} W$ whose composition is an isomorphism. This implies that W is a summand of U, hence is isomorphic to U by g because U is indecomposable, and so f is split mono.

2. is proved similarly.

3. We see that $f \in \mathrm{Rad}_A^2(U, V)$ if and only if f factors as $f = hg$ with $g \in \mathrm{Rad}_A(U, W)$ and $h \in \mathrm{Rad}_A(W, V)$, for some module W. By parts 1. and 2. it is equivalent to say that g is not split mono and h is not split epi. Equivalently, f is not an isomorphism and is not irreducible. $\qquad\square$

Definition 5.3.7. For each pair of indecomposable A-modules U and V we define $\mathrm{Irr}(U, V) := \mathrm{Rad}_A(U, V)/\mathrm{Rad}_A^2(U, V)$. This vector space is called the *space of irreducible morphisms* from U to V, although its elements are not morphisms.

We see that $\mathrm{Irr}(U, V)$ is a left $\mathrm{End}_A(V)$-module and a right $\mathrm{End}_A(U)$-module. The terminology for this is that it is a $(\mathrm{End}_A(V), \mathrm{End}_A(U))$-*bimodule*. It is annihilated by both $\mathrm{Rad}(\mathrm{End}_A V)$ and by $\mathrm{Rad}(\mathrm{End}_A U)$, so that it is a bimodule for the division ring quotients D_V and D_U of these endomorphism rings.

We make the following definition for A-mod where A is a finite dimensional algebra, but it applies to Krull-Schmidt categories generally.

Definition 5.3.8. The *Auslander-Reiten quiver* of the finite dimensional algebra A is the directed graph with the isomorphism classes $[U]$ of indecomposable modules U as vertices. There is an arrow $[U] \xrightarrow{(d_{UV}, d^{UV})} [V]$ from $[U]$ to $[V]$, where $d_{UV} = \dim_{D_V} \mathrm{Irr}(U, V)$ and $d^{UV} = \dim_{D_U} \mathrm{Irr}(U, V)$. If $d_{UV} = d^{UV}$ we may omit the labeling and replace the labeled edge by d_{UV} unlabeled edges.

When the ground field K is algebraically closed we must always have that the division rings $D_U = D_V = K$ and $d_{UV} = d^{UV}$.

Example 5.3.9. When $A = K[x]/(x^n)$ there is an indecomposable module V_i of each dimension up to n. There are monomorphisms $V_i \to V_{i+1}$ and epimorphisms $V_{i+1} \to V_i$, and these are (representatives of) all the irreducible maps between these modules. In fact, the irreducible morphisms $V_i \to V_j$ (when there are any) form a single orbit under the the unit groups of $\mathrm{End}_A(V_i)$ and of $\mathrm{End}_A(V_j)$. From this we see that

$$d_{V_i V_{i+1}} = d^{V_i V_{i+1}} = d_{V_{i+1} V_i} = d^{V_{i+1} V_i} = 1 \text{ when } 1 \leq i \leq n - 1.$$

The Auslander-Reiten quiver is

$$V_1 \overset{\rightarrow}{\leftarrow} V_2 \overset{\rightarrow}{\leftarrow} \cdots \overset{\rightarrow}{\leftarrow} V_n.$$

It is clear that $\dim \mathrm{Irr}(U, V)$ is finite when U and V are indecomposable modules, so that sets of irreducible morphisms $U \to V$ whose images in $\mathrm{Irr}(U, V)$ are independent must necessarily be finite. What is not clear at this stage is whether there can be found infinitely many non-isomorphic indecomposable modules W with an irreducible morphism $U \to W$. This is the same as asking whether the functor $\mathrm{Rad}(U, -)$ is finitely generated or, equivalently, whether the simple functors S_U are finitely presented. We will see an answer to this question as a consequence of the existence of Auslander-Reiten sequences.

5.4 Auslander-Reiten sequences

Given a short exact sequence of modules $0 \to U \overset{\alpha}{\to} V \overset{\beta}{\to} W \to 0$ we consider the following conditions:

(1_L) U is indecomposable,

(1_R) W is indecomposable,

(2) the sequence does not split,

(2_L) α is not split mono,

(2_R) β is not split epi,

(3_L) give any morphism $\rho : U \to U'$ that is not split mono, there exists a morphism $\sigma : V \to U'$ so that $\rho = \sigma \alpha$,

(3_R) give any morphism $\theta : W' \to W$ that is not split epi, there exists a morphism $\phi : W' \to V$ so that $\theta = \beta \phi$,

(4_L) whenever $h \in \mathrm{End}(V)$ is such that $h\alpha = \alpha$ then h is an automorphism,

(4_R) whenever $h \in \mathrm{End}(V)$ is such that $\beta h = \beta$ then h is an automorphism,

(5_L) α is irreducible,

(5_R) β is irreducible,

Definition 5.4.1. If α satisfies 4_L it is called *left minimal*, and if β satisfies 4_R it is called *right minimal*. If α satisfies 2_L and 3_L it is called *left almost split*, and if β satisfies 2_R and 3_R it is called *right almost split*. If α is minimal and left almost split it is called *left minimal almost split*, and if β is minimal and right almost split it is called *right minimal almost split*.

These conditions are not independent. We already know that $(2) \Leftrightarrow (2_L) \Leftrightarrow (2_R)$. Other implications, such as $(3_L) \Rightarrow (1_L)$, $(3_R) \Rightarrow (1_R)$, are less obvious. Notice also that if α were split mono then (3_L) would hold (without requiring that ρ be not split mono), and similarly if β were split epi then (3_R) would hold.

Definition 5.4.2. We say that the short exact sequence $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ is an *almost split sequence*, or an *Auslander-Reiten sequence*, if α is left minimal almost split and β is right minimal almost split.

Theorem 5.4.3. *Let U, V and W be finite dimensional modules for a finite dimensional algebra. The following are equivalent for a short exact sequence $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$.*

1. *α is left almost split and β is right almost split.*

2. *α is left minimal almost split.*

3. *β is right minimal almost split.*

4. *U is indecomposable and β is right almost split.*

5. *W is indecomposable and α is left almost split.*

6. *α and β are irreducible.*

Over a finite dimensional algebra, all of the conditions listed in Theorem 5.4.3 are equivalent to the definition of an almost split sequence, and in this context there is redundancy in the definition we have given of an almost split sequence. Some of the conditions are more convenient to use than others. Conditions 4. or 5. often turn out to work well with in practice. Condition 6. may be less helpful, because we do not have much idea at this stage what the irreducible morphisms look like.

Parts of the equivalences in the above result will come from the following proposition.

Proposition 5.4.4. *Let $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ be a short exact sequence.*

1. *If α is left almost split then U is indecomposable.*

2. *If α is minimal left almost split then α is irreducible.*

3. *If α is irreducible then W is indecomposable.*

4. *For finite dimensional modules, if W is indecomposable and the sequence does not split then α is left minimal.*

Analogous statements hold regarding β.

Proof. 1. Suppose that $U = U_1 \oplus U_2$ is the direct sum of two non-zero submodules. Then, for each i, projection $p_i : U \to U_i$ is not split mono, so there exists $v_i : V \to U_i$ so that $p_i = v_i \alpha$. We deduce that α is split by the map $V \to U$ with components $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$, which is not possible. Thus U is indecomposable.

2. Suppose that α is minimal left almost split. Then α is not split mono, and it is not split epi because then α would be an isomorphism. Assume that $\alpha = \alpha_1 \alpha_2$ where $\alpha_2 : U \to X$ and $\alpha_1 : X \to V$. We suppose that α_2 is not split mono and show that α_1 must be split epi. Because α is left almost split, there exists $\alpha_2' : V \to X$ so that $\alpha_2 = \alpha_2' \alpha$. Hence $\alpha = \alpha_1 \alpha_2 = \alpha_1 \alpha_2' \alpha$. Because α is left minimal, $\alpha_1 \alpha_2'$ is an automorphism and so α_1 is split epi.

3. Suppose that α is irreducible and that $W = W_1 \oplus W_2$ is the direct sum of two non-zero submodules. For each i let $q_i : W_i \to W$ be the inclusion and consider the commutative (pullback) diagram with exact rows

$$
\begin{array}{ccccccccc}
0 & \to & U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \to & 0 \\
 & & \| & & u \uparrow & & q_i \uparrow & & \\
0 & \to & U & \xrightarrow{\alpha'} & V_i & \xrightarrow{\beta'} & W_i & \to & 0
\end{array}
$$

where V_i is the preimage in V of W_i. Now $\alpha = u\alpha'$ and u is not split epi, so α' is split mono and hence β' is split epi. Thus there exists $v_i : W_i \to V_i$ so that $\beta' v_i = 1_{W_i}$ and now $uv_i : W_i \to V$ has the property that $q_i = \beta uv_i$. This means the map $W \to V$ with components $[uv_1, uv_2]$ splits β. We deduce that α is split mono, which is not possible for an irreducible morphism. Thus W is indecomposable.

4. Suppose that W is indecomposable and that $h : V \to V$ satisfies $h\alpha = \alpha$, so that h is the identity on $\alpha(U)$. Then h induces a map $g : W \to W$. If g is an isomorphism, so is h, and we are done. Otherwise g is not an isomorphism so that $g^n = 0$ for some n because W is indecomposable. Now $h^n \alpha = \alpha$ and $h^n : V \to U$ has image equal to $\alpha(U)$, where it is the identity. This means h^n provides a splitting of the sequence, which does not happen. Thus h is an isomorphism. $\qquad\square$

Class Activity. In the next proof, determine which part or parts of Proposition 5.4.4 are used in the implications 1. implies 4. and 5, 5. implies 2, 4. and 5. imply 6. and 6. implies that U and W are indecomposable.

Proof of Theorem 5.4.3. We see from Proposition 5.4.4 that 1. implies 4. and 5; that 5. implies 2. and 4. implies 3. for finite dimensional algebras; and that 4. and 5. taken together imply 6, which implies that U and W are indecomposable. We show that 3. implies 1, and from this and the similar 4. implies 1. we conclude that the first 5 conditions are equivalent. We omit 6. implies 1: see Auslander-Reiten-SmaløIV.1.13.

To show that 3. implies 1, suppose that β is right minimal almost split. Let $u : V \to X$ be a module homomorphism with $u \neq u'\alpha$ for all $u' : V \to X$. We show

that u is split mono. We construct the pushout commutative diagram

$$
\begin{array}{ccccccccc}
0 & \to & U & \xrightarrow{\alpha} & V & \xrightarrow{\beta} & W & \to & 0 \\
& & \downarrow{u} & & \downarrow{v} & & \| & & \\
0 & \to & X & \xrightarrow{\gamma} & Y & \xrightarrow{\delta} & W & \to & 0
\end{array}
$$

where $Y := (X \oplus V)/\{(u(z), -\alpha(z)) \mid z \in U\}$. Now γ is not split mono, otherwise the splitting precomposed with v would provide a map u' with $u = u'\alpha$, which was excluded. Therefore δ is not split epi. Because β is right almost split, δ lifts to a mapping $\bar{v} : Y \to V$ so that $\delta = \beta\bar{v}$. Thus $\beta = \delta v = \beta\bar{v}v$. From right minimality of β we deduce that $\bar{v}v$ is an automorphism. Thus $\bar{v}$ restricts to a map $\bar{u} : X \to U$ with $\bar{u}u$ an automorphism. This means that u is split mono. $\qquad\square$

We now list some first properties of almost split sequences.

Proposition 5.4.5. *Let $E : 0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ be an almost split sequence.*

1. *Every almost split sequence $E' : 0 \to U \xrightarrow{\alpha'} V' \xrightarrow{\beta'} W' \to 0$ starting with U is isomorphic to E.*

2. *Every almost split sequence $E'' : 0 \to U'' \xrightarrow{\alpha''} V'' \xrightarrow{\beta''} W \to 0$ ending with W is isomorphic to E.*

Proof. We only prove 1. Because α and α' are not split mono there exist maps $\sigma : V \to V'$ and $\sigma' : V' \to V$ so that $\alpha' = \sigma\alpha$ and $\alpha = \sigma'\alpha'$. Thus $\alpha = \sigma'\sigma\alpha$ and $\alpha' = \sigma\sigma'\alpha'$. Left minimality of α implies that σ and σ' are automorphisms. They extend to commutative diagrams showing that the two sequences are isomorphic. $\qquad\square$

Apart from anything else, this means that, up to isomorphism, the left hand module of an almost split sequence is determined by the right hand module, and vice-versa.

Proposition 5.4.6. *A short exact sequence of left modules*

$$
0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0
$$

for a finite dimensional algebra A is an almost split sequence if and only if

$$
0 \to D(W) \xrightarrow{D(\beta)} D(V) \xrightarrow{D(\alpha)} D(U) \to 0
$$

is an almost split sequence of right A-modules.

Proof. This is immediate from the fact that D is a duality. $\qquad\square$

Proposition 5.4.7. *Let A be a finite dimensional algebra.*

1. *Let $0 \to U \xrightarrow{\alpha} V \xrightarrow{\beta} W \to 0$ be an almost split sequence. A morphism $X \xrightarrow{\gamma} W$ is irreducible if and only if X is isomorphic to a summand of V and $\gamma = \beta \circ i$ where i is the inclusion of X as a summand. A morphism $U \xrightarrow{\delta} Y$ is irreducible if and only if Y is isomorphic to a summand of V and $\delta = p \circ \alpha$ where p is projection onto Y as a summand.*

2. *If P is an indecomposable projective A-module, a morphism $\gamma : X \to P$ is irreducible if and only if γ is an inclusion of X as a summand of $\mathrm{Rad}\, P$. If I is an indecomposable injective A-module, a morphism $\delta : I \to Y$ is irreducible if and only if δ is a projection of $I/\mathrm{Soc}\, I$ onto Y as a summand.*

Proof. 1. We only prove the first part. If $\gamma : X \to W$ is irreducible then γ is not split epi so lifts to a morphism $u : X \to V$ so that $\gamma = \beta \circ u$. Now β is not split epi, so u must be split mono, and embeds X as a summand of V. We leave as an exercise the fact that if X is a summand of V then the restriction of β to X is irreducible.

2. We have already seen that the inclusion $\mathrm{Rad}\, P \to P$ is irreducible. We leave the rest as an exercise. $\qquad\square$

We conclude in all cases that if W is an indecomposable A-module that is either projective or the right hand term of an almost split sequence, then there are only finitely many isomorphism classes of indecomposable modules X for which there is an irreducible morphism $X \to W$. A similar statement holds for irreducible morphisms $U \to Y$ when U is indecomposable.

Theorem 5.4.8. *Let $0 \to U \to \bigoplus_i V_i^{n_i} \to W$ be an almost split sequence, where the summands V_i of the middle term have multiplicity n_i, and $V_i \not\cong V_j$ if $i \neq j$. Then $n_i = \dim_{D_W} \mathrm{Irr}(V_i, W) = \dim_{D_U} \mathrm{Irr}(U, V_i)$, where D_U and D_W are the radical quotient division rings of $\mathrm{End}(U)$ and $\mathrm{End}(W)$.*

Proof. See Assem-Simson-Skowronski IV.4.2. $\qquad\square$

5.5 The Auslander-Reiten translate

We define the transpose of a module and start by approximating each left A-module W by projective modules.

Definition 5.5.1. Let
$$P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} W \to 0$$
be a *minimal projective presentation* of W. This means that the sequence is exact and both $p_0 : P_0 \to W$ and $p_1 : P_1 \to \mathrm{Ker}\, p_0$ are projective covers. Applying the left exact contravariant functor $(-)^\vee$ we obtain an exact sequence of right A-modules
$$0 \to W^\vee \to P_0^\vee \to P_1^\vee \to \mathrm{Tr}\, W \to 0$$
where $\mathrm{Tr}\, W$ is defined as the cokernel of $p_1^\vee$.

Example 5.5.2. Let Q be the quiver $1 \to 2 \leftarrow 3$, with indecomposable projective left modules $P_1 = \frac{1}{2}$, $P_2 = {}_2$, $P_3 = \frac{3}{2}$. We compute $\mathrm{Tr}({}^1 {}_2 {}^3)$ by taking a minimal projective presentation

$$P_2 \to P_1 \oplus P_3 \to {}^1 {}_2 {}^3 \to 0$$

Applying $(-)^\vee$ we get right modules, which are modules for the opposite quiver $1 \leftarrow 2 \to 3$ and we get an exact sequence of right modules

$$P_1 \oplus P_3 \to P_2 \to \mathrm{Tr}({}^1 {}_2 {}^3) \to 0.$$

From this we deduce that $\mathrm{Tr}({}^1 {}_2 {}^3) = {}_2$.

Class Activity. For $Q = 1 \to 2 \leftarrow 3$ compute $D\,\mathrm{Tr}(1)$.

From what we know so far, we can say that Tr is only defined on modules up to isomorphism. The isomorphism type of $\mathrm{Tr}\,W$ is uniquely determined because projective covers of modules are uniquely determined up to isomorphism. We can ask whether Tr is defined as a functor, and it is, but on categories that factor out the isomorphisms that can occur. If we use the stable category associated to A-mod, in which morphisms are cosets of the set of morphisms that factor through projective modules, then Tr provides an equivalence between the stable category of A-mod and the stable category of mod-A. Without going into such technicalities, we have the following:

Proposition 5.5.3. *Let W be an indecomposable left A-module.*

1. *The right A-module $\mathrm{Tr}\,W$ has no nonzero projective direct summands.*

2. *If W is not projective, then the sequence $P_0^\vee \to P_1^\vee \to \mathrm{Tr}\,W \to 0$ induced from the minimal projective presentation $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} W \to 0$ of W is a minimal projective presentation of $\mathrm{Tr}\,W$.*

3. *W is projective if and only if $\mathrm{Tr}\,W = 0$. If W is not projective, then $\mathrm{Tr}\,W$ is indecomposable and $\mathrm{Tr}(\mathrm{Tr}\,W) \cong W$.*

4. *If W and X are indecomposable nonprojective, then $W \cong X$ if and only if $\mathrm{Tr}\,W \cong \mathrm{Tr}\,X$.*

5. *Tr provides a bijection between isomorphism classes of non-projective indecomposable left A-modules, and isomorphism classes of non-projective indecomposable right A-modules.*

Proof. See Assem-Simson-Skowronski IV.2.1. We give a brief version.

1. If $\mathrm{Tr}\,W = Q \oplus V$ where Q is projective, then Q splits off $P_1^\vee$ so that $P_1^\vee \cong Q \oplus X$ for some module X containing the image of $P_0^\vee \to P_1^\vee$. It follows that $P_1 \cong P_1^{\vee\vee} \cong Q^\vee \oplus X^\vee$ and $X^\vee$ is a summand of $P_1^{\vee\vee}$ contained in the kernel of $P_1^{\vee\vee} \to P_0^{\vee\vee}$. This implies that the projective presentation of W was not minimal.

2. If it is not a minimal projective presentation of $\mathrm{Tr}\,W$ then one of $P_0^\vee$ and $P_1^\vee$ has a direct summand that is mapped to zero. Applying $(-)^\vee$ we see that the original projective presentation was not minimal.

3. If W is projective then $\operatorname{Tr} W = 0$ because in the minimal projective presentation of W we can take $P_0 = W$ and $P_1 = 0$. If $\operatorname{Tr} W = 0$ then $P_0^\vee \to P_1^\vee$ is epi, and hence split because $P_1^\vee$ is projective. We deduce that $P_1 \to P_0$ is split mono, $P_1 = 0$ by minimality, and $P_0 \cong W$ is projective.

If W is not projective we deduce that $\operatorname{Tr}(\operatorname{Tr} W) \cong W$ by applying 2. twice. If $\operatorname{Tr} W = W_1 \oplus W_2$ with nonzero summands then $W \cong \operatorname{Tr}(W_1) \oplus \operatorname{Tr}(W_2)$ with neither summand 0, which cannot happen, so that $\operatorname{Tr} W$ is indecomposable.

4. If $W \cong X$ then $\operatorname{Tr} W \cong \operatorname{Tr} X$, which in turn implies $W \cong \operatorname{Tr}\operatorname{Tr} W \cong \operatorname{Tr}\operatorname{Tr} X \cong X$.

5. is now immediate. $\qquad\square$

Definition 5.5.4. The *Auslander-Reiten translate* is defined on A-modules as $\tau = D\operatorname{Tr}$. Thus if W is a left A-module then $\tau W = D\operatorname{Tr} W$. We also define $\tau^{-1} = \operatorname{Tr} D$.

Corollary 5.5.5. *The Auslander-Reiten translate τ provides a bijection*

{isomorphism classes of indecomposable non-projective A-modules}

$\rightarrow$ *{isomorphism classes of indecomposable non-injective A-modules}*

with the inverse bijection provided by τ^{-1}.

The Nakayama functor $\nu = D((-)^\vee)$ provides a good way to compute τ.

Proposition 5.5.6. *1. Let $P_1 \xrightarrow{p_1} P_0 \xrightarrow{p_0} W \to 0$ be a minimal projective presentation of W. Then there is an exact sequence*

$$0 \to \tau W \to \nu P_1 \to \nu P_0 \to \nu W \to 0.$$

2. Let $0 \to U \to I_0 \xrightarrow{\iota_0} I_1 \xrightarrow{\iota_1}$ be a minimal injective presentation of U. Then there is an exact sequence

$$0 \to \nu^{-1} U \to \nu I_0 \to \nu I_1 \to \tau^{-1} U \to 0.$$

Proof. We get this by applying the exact functor D to the sequence

$$0 \to W^\vee \to P_0^\vee \to P_1^\vee \to \operatorname{Tr} W \to 0.$$

The second part is proved similarly. $\qquad\square$

Example 5.5.7. We calculate τ for the indecomposable modules for the quiver $1 \to 2 \to 3$.

5.6   Existence of Auslander-Reiten sequences

It is quite surprising that, for modules over a finite dimensional algebra, almost split sequences exist. In view of Proposition 5.4.7 it has the implication that, for each indecomposable module W, there are only finitely many isomorphism classes of modules V for which there is an irreducible morphism $V \to W$ and, for each indecomposable module U, there are only finitely many isomorphism classes of modules V for which there is an irreducible morphism $U \to V$.

Theorem 5.6.1. *Let A be a finite dimensional algebra.*

1. *For each indecomposable non-projective A-module W there exists an almost split sequence $0 \to \tau W \to V \to W \to 0$.*

2. *For each indecomposable non-injective A-module U there exists an almost split sequence $0 \to U \to V \to \tau^{-1}U \to 0$.*

Proof. The proof uses constructions from homological algebra, including a remarkable duality isomorphism. These constructions go beyond what we have done so far. At the end of the proof we realize that a construction has been made, but it does not help us too much to understand the properties of almost split sequences. We sketch what happens.

The proof starts with the Auslander-Reiten duality formula. Let W be an indecomposable non-projective A-module. We have

$$D\underline{\mathrm{Hom}}_A(L, W) \cong \mathrm{Ext}^1_A(W, \tau L)$$

valid for any module L, where $\underline{\mathrm{Hom}}_A(L, W)$ is the quotient of $\mathrm{Hom}_A(L, W)$ by the homomorphisms that factor through a projective module. Taking $L = W$ we get a quotient of the local ring $\mathrm{End}_A(W)$, which has a simple radical quotient as a module for $\mathrm{End}_A(W)$. It follows that $\mathrm{Ext}^1_A(W, \tau W)$ has a simple socle as a left $\mathrm{End}_A(W)$-module and also as a right $\mathrm{End}_A(W)$-module. This Ext group classifies module extensions of W by τW, and a non-zero element in the socle turns out to correspond to an almost split sequence. The two end terms are indecomposable, the sequence is not split and the right almost split property is satisfied: any element of $\mathrm{Rad}(X, W)$ acts as zero on the extension, so produces a split extension under pullback. This provides a lifting as required for the right almost split property. $\qquad\square$

There are some situations where we can immediately write down almost split sequences.

Proposition 5.6.2. *Let P be an indecomposable module that is both projective and injective, and not simple. Then the sequence*

$$0 \to \mathrm{Rad}\, P \to (\mathrm{Rad}\, P/\mathrm{Soc}\, P) \oplus P \to P/\mathrm{Soc}\, P \to 0$$

is almost split, where the component maps are the inclusions and projections (with a minus sign, as appropriate).

Example 5.6.3. The quiver $1 \to 2 \to 3$ has a projective-injective indecomposable module. An algebra is *self-injective* if all projective modules are injective (and all injective modules are projective).

Proof. The module P is the projective cover of a simple module S, being indecomposable projective, and the injective envelope of a simple module T, being indecomposable injective. Thus $P = P_S = I_T$. Notice that $\operatorname{Rad} P_S$ and $P_S / \operatorname{Soc} P_S$ are both indecomposable in this situation. We compute $D \operatorname{Tr}(P_S / \operatorname{Soc} P_S)$ using the projective presentation

$$P_T \xrightarrow{p_1} P_S \to P_S / \operatorname{Soc} P_S \to 0.$$

Applying the Nakayama functor ν gives an exact sequence

$$0 \to D \operatorname{Tr}(P_S / \operatorname{Soc} P_S) \to I_T \xrightarrow{\nu p_1} I_S.$$

We show that νp_1 has image $S = \operatorname{Soc} I_S$, which will imply that $D \operatorname{Tr}(P_S / \operatorname{Soc} P_S) = \operatorname{Rad} P_S$. This can be seen because p_1 has the property that for all $f : P_S \to Q$ where Q is indecomposale projective, either f is an isomorphism or $f p_1 = 0$. Therefore, for all $g : I_S \to J$ where J is injective indecomposable, either g is an isomorphism or $g \nu p_1 = 0$. If the image of νp_1 is larger than S, we can find a non-isomorphism $g : I_S \to J$ so that $g \nu p_1 \neq 0$, which cannot happen, so we deduce that νp_1 has image S.

We have now shown that $\tau(P_S / \operatorname{Soc} P_S) = \operatorname{Rad} P_S$, so there is an almost split sequence

$$0 \to \operatorname{Rad} P_S \to V \oplus P_S \to P_S / \operatorname{Soc} P_S \to 0$$

for some module V, using the fact that there are irreducible morphisms $\operatorname{Rad} P_S \to P_S$ and $P_S \to P_S / \operatorname{Soc} P_S$. Counting composition factors, V has the same composition factors as $\operatorname{Rad} P_S / \operatorname{Soc} P_S$. This means the component irreducible morphism $\operatorname{Rad} P_S \to V$ cannot be a monomorphism. Therefore it is an epimorphism, and V is a proper quotient of $\operatorname{Rad} P_S$, which has a simple socle. Thus V is a quotient of $\operatorname{Rad} P_S / \operatorname{Soc} P_S$ with the same composition factors, so $V \cong \operatorname{Rad} P_S / \operatorname{Soc} P_S$ and the component irreducible morphism from $\operatorname{Rad} P_S$ is the quotient map. Similarly, the map $V \to P_S / \operatorname{Soc} P_S$ is the inclusion to $\operatorname{Rad} P_S / \operatorname{Soc} P_S$. $\qquad \square$

5.7 Calculations

Example 5.7.1. We calculate the Auslander-Reiten quiver of the quivers $1 \to 2 \to 3$ and $1 \to 2 \to 3 \to 4$. We do this by writing down all the irreducible morphisms that arise as inclusions of radicals of projectives, and as quotients maps to quotients of injectives by their socles. We also write down the sequence for the projective-injective indecomposable. For $1 \to 2 \to 3$ this gives all the irreducible morphisms. For $1 \to 2 \to 3 \to 4$ we complete the quiver from counting the composition factors of modules in the middle term of an almost split sequence.

Example 5.7.2. We calculate the Auslander-Reiten quiver of the incidence algebra of the poset with Hasse diagram
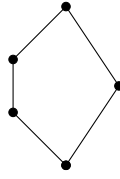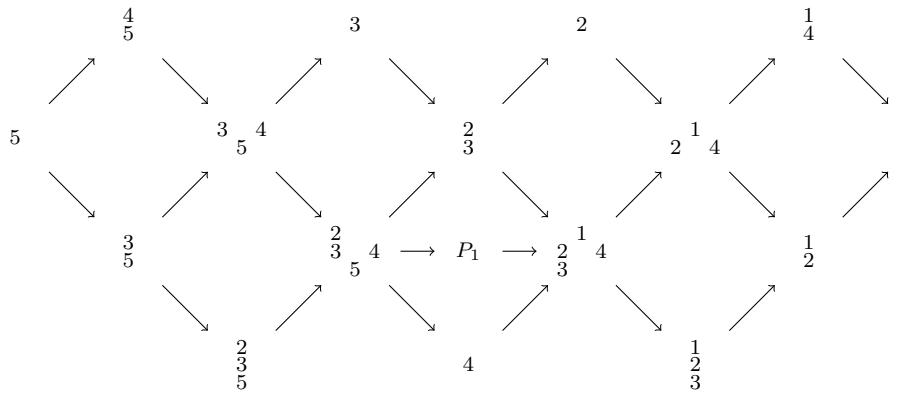
over an arbitrary field. We take the elements of the poset to be $\{1, 2, 3, 4, 5\}$ with partial order given by $1 < 2 < 3 < 5$ and $1 < 4 < 5$. The indecomposable projective modules are

$$P_1 = {\substack{1 \\ 2 \; 4 \\ 3 \\ 5}}, \quad P_2 = {\substack{2 \\ 3 \\ 5}}, \quad P_3 = {\substack{3 \\ 5}}, \quad P_4 = {\substack{4 \\ 5}}, \quad P_5 = 5$$

and the indecomposable injective modules are

$$I_1 = 1, \quad I_2 = {\substack{1 \\ 2}}, \quad I_3 = {\substack{1 \\ 3}}, \quad I_4 = {\substack{1 \\ 4}}, \quad I_5 = {\substack{1 \\ 2 \; 4 \\ 3 \\ 5}}$$

The Auslander-Reiten quiver is

We calculate

- The AS sequence with the indecomposable projective-injective $P_1 = I_5$ in the middle,

- $D\operatorname{Tr}(4)$ and $D\operatorname{Tr}({\substack{2 \\ 3}})$. From this we deduce that the AS sequence with 4 at the right has indecomposable middle term, and the AS sequence with ${\substack{2 \\ 3}}$ at the right has 3 as a summand of its middle term.

- The inclusion of the radical $P_3 \to P_2$, which shows that $P_3 = D\operatorname{Tr}({\substack{2 \\ 3 \; 4 \\ 5}})$, because P_3 is not injective, and there is only one irreducible morphism out of P_2.

- We place the irreducible morphism $P_5 \to P_4$, finding that $P_5 = D\operatorname{Tr}({\substack{3 \\ 5 \; 4}})$, because there are only two irreducible morphisms out of P_3, and that $P_4 = D\operatorname{Tr}(3)$.

- We complete the right half of the quiver by duality.

- We have constructed a connected component of the quiver. For all we know there by further components that we have not yet discovered. By a theorem of Auslander, if the quiver has a finite connected component, and the algebra cannot be written as a direct sum of ideals, then the quiver is complete.

Notice that only two direct $\tau = D\operatorname{Tr}$ calculations were done, and that the remaining $D\operatorname{Tr}$ values were deduced from the structure of the quiver. Notice also that the equivalence A-proj $\simeq A$-inj given by the Nakayama functor does not preserve irreducible morphisms.

Example 5.7.3. We calculate a component of the Auslander-Reiten quiver of the Kronecker quiver $1 \Longrightarrow 2$ A representation of this quiver is the specification of a pair of matrices from one vector space to another, and isomorphism of such representations corresponds to simultaneous conjugacy of the matrices by base change matrices for the two spaces. The problem of finding a normal form for such pairs of matrices was solved by Kronecker, and corresponds to the classification of indecomposable representations of the quiver. For this reason the quiver is known as the Kronecker quiver.

The indecomposable projective and injective modules have the form

$$P_1 = {}_2 1_2, \quad P_2 = {}_2, \quad I_1 = {}_1, \quad I_2 = {}^1 {}_2 {}^1.$$

We calculate $D\operatorname{Tr}(1)$ by taking the minimal projective resolution $P_2^2 \to P_1 \to 1 \to 0$ and applying ν to get an exact sequence $0 \to D\operatorname{Tr}(1) \to I_2^2 \to I_1$ from which we get $D\operatorname{Tr}(1) = {}^1 {}_2 {}^1 {}_2 {}^1$. This provides a proof that the last module is indecomposable. (Calculation also shows that ${}^1 {}_2 {}^1 {}_2 {}^1 \cong \mathcal{R}_1^+ \mathcal{R}_2^+(1)$, which again provides a proof that this module is indecomposable.)

There is thus an almost split sequence

$$0 \to {}^1 {}_2 {}^1 {}_2 {}^1 \to E \to {}_1 \to 0$$

for some module E. We know by Proposition 5.4.7 that there are two irreducible morphisms $I_2 = {}^1 {}_2 {}^1 \to {}_1$, so that E must have two copies of I_2 as a direct summand. By counting composition factors we see that $E \cong I_2^2$ and the almost split sequence is

$$0 \to {}^1 {}_2 {}^1 {}_2 {}^1 \to {}^1 {}_2 {}^1 \oplus {}^1 {}_2 {}^1 \to {}_1 \to 0.$$

Continuing in this fashion we compute $D\operatorname{Tr}({}^1 {}_2 {}^1) = {}^1 {}_2 {}^1 {}_2 {}^1$ so that there is an almost split sequence

$$0 \to {}^1 {}_2 {}^1 {}_2 {}^1 {}_2 {}^1 \to {}^1 {}_2 {}^1 {}_2 {}^1 \oplus {}^1 {}_2 {}^1 {}_2 {}^1 \to {}^1 {}_2 {}^1 \to 0.$$

In this way we construct a quiver component that extends infinitely to the left. It does not extend to the right because I_1 and I_2 are injective.

By dual calculations with $\tau^{-1} = \operatorname{Tr} D$ we obtain almost split sequences

$$0 \to {}_2 \to {}_2 {}^1 {}_2 \oplus {}_2 {}^1 {}_2 \to {}_2 {}^1 {}_2 {}^1 {}_2 \to 0$$

and

$$0 \to {}_2{}^1{}_2 \to {}_2{}^1{}_2{}^1{}_2 \oplus {}_2{}^1{}_2{}^1{}_2 \to {}_2{}^1{}_2{}^1{}_2{}^1{}_2 \to 0$$

and so on, constructing a quiver component that starts with the indecomposable projectives at the left and extends infinitely to the right. These quiver components have the form of a semi-infinite graph along the real line with vertices at either negative integers or positive integers, and with edges labeled $(2, 2)$.

The indecomposable modules just constructed are known as the *string modules* for this quiver. It is a fact that we have now seen a complete list of the string modules, and they account for all the indecomposables whose total dimension is odd. There remain infinitely many further Auslander-Reiten quiver components for the Kronecker quiver.

Chapter 6

Representation type

We follow section IV.5 of Assem-Simson-Skowronski for applications to finite representation type and Brauer-Thrall questions.

Chapter 7

Functorial methods

We follow section IV.6 of Assem-Simson-Skowronski for a discussion of representations of the category of A-modules, including the correspondence between AR sequences and projective resolutions of the simple functors, and Auslander algebras.

We let A be a finite dimensional K-algebra and let $\operatorname{Fun} A$ denote the category of K-linear functors A-mod $\to K$-mod. The morphisms between such functors are the natural transformations that are k-linear maps at each object. The theory of these functors is very similar to the theory of all functors from a category to the category of vector spaces, that we have considered before, but there are some differences. Such linear functors

- preserve direct sums,

- are determined by their values on indecomposable objects and the morphisms between them,

so that $\operatorname{Fun} A$ is equivalent to the category of K-linear functors A-ind $\to K$-mod where A-ind is the full subcategory of A-mod whose objects are (a set of representatives of isomorphism classes of) the indecomposable A-modules. Morphisms in $\operatorname{Fun} A$ are mono or epi if and only if at each evaluation they are mono or epi. A sequence of natural transformations of functors is exact at some functor if and only if on each evaluation the sequence is exact.

If A has finite representation type and we take a set of representatives $U_1, \ldots, U_t$ of the indecomposable A-modules we may form a finite dimensional algebra that is the vector space $B = \bigoplus_{i,j} \operatorname{Hom}_A(U_j, U_i)$. The product in B of elements that lie in these summands is defined to be composition if they can be composed, and 0 otherwise.

Proposition 7.0.1. *Let A have finite representation type. The algebra B just defined is isomorphic to $\operatorname{End}_A(\bigoplus_i U_i)$.*

Proof. This is a matter of examining the definitions and using the fact that every endomorphism of a direct sum can be written as a matrix of component morphisms between the summands. $\square$

Definition 7.0.2. Let A have finite representation type with representatives $U_1, \ldots, U_t$ of the indecomposable A-modules. The algebra $\operatorname{End}_A(\bigoplus_i U_i)$ is called the *Auslander algebra* of A.

The Auslander algebra is defined up to isomorphism, depending on the choice of representatives of the indecomposable A-modules.

Proposition 7.0.3. *Let A have finite representation type and let B be the Auslander algebra of A. Then* Fun A *and B-mod are equivalent categories.*

Proof. Given a functor $F : A\text{-mod} \to K\text{-mod}$ we get a B module $\bigoplus_i F(U_i)$. Given a B-module M we get a functor with $F(U) := 1_U \cdot M$. These two functors provide inverse equivalences. $\qquad\square$

When A is of infinite representation type the construction described above does not produce a finite dimensional algebra. We still get a vector space with a product defined, but it has no identity. In this situation we could work with such an algebraic structure, but we do not get an equivalence of categories as above, and it is more elegant to work with functors.

Exercise 7.0.4. Consider the Auslander algebra B of the path algebra of the quiver $1 \to 2$ over some field. Find its dimension and the structure of its projective and injective modules. Compute its Auslander-Reiten quiver. Show that submodules of projective modules need not be projective.

We have seen before with representations of a category $\mathcal{C}$ that for each object x of $\mathcal{C}$ there is a projective $K\mathcal{C}$-module $K\mathcal{C} \cdot 1_x$, and that it corresponds to the linearization of a representable functor: $K \operatorname{Hom}_{\mathcal{C}}(x, -)$ that, at each object y, returns the vector space with basis $\operatorname{Hom}_{\mathcal{C}}(x, y)$. There is something similar for linear functors from linear categories. We now do IV.6.1 - IV.6.11 of Assem-Simson-Skowronski.

Definition 7.0.5. Let $\mathcal{C}$ be a K-linear category and x and object of $\mathcal{C}$. A functor $M : \mathcal{C} \to K\text{-mod}$ is called *representable at x* if M is naturally isomorphic to the functor $F_x := \operatorname{Hom}_{\mathcal{C}}(x, -)$. It is *representable* if it is representable at x for some x.

Theorem 7.0.6 (Linearized Yoneda Lemma)**.** *Let $\mathcal{C}$ be a K-linear category and x an object of $\mathcal{C}$. For any K-linear functor $M \to K$-mod there is a natural bijection $\operatorname{Hom}_{\mathcal{C}}(F_x, M) \leftrightarrow M(x)$.*

Proof. The bijection is given as follows: given a natural transformation $\eta : F_x \to M$ we map it to the element $\eta_x(1_x) \in M(x)$ and this defines a mapping from left to right. Given an element $a \in M(x)$ we map it to the natural transformation $\theta : F_x \to M$ where, at an object y, we have $\theta_y(\phi) = M(\phi)(a) \in M(y)$. Here $\phi \in F_x(y)$ is a morphism from x to y. We must check various things: this does define a natural transformation θ, and the two mappings are inverse and natural. $\qquad\square$

Corollary 7.0.7. *Let x and y be objects in a K-linear category $\mathcal{C}$.*

1. *There is an isomorphism of K-vector spaces $\operatorname{Hom}(F_y, F_x) \cong \operatorname{Hom}_{\mathcal{C}}(x, y)$. Every natural transformation $F_y \to F_x$ has the form α^* where $\alpha : x \to y$ and α^* is composition with α.*

2. *$F_x \cong F_y$ if and only if $x \cong y$.*

3. *F_x is a projective object in the category of K-linear functors $\mathcal{C} \to K$-mod.*

Proof. 1. The isomorphism of vector spaces is immediate. To see that every natural transformation $F_y \to F_x$ has the form α^* we examine the construction in the proof of Yoneda's Lemma. Given $\alpha : x \to y$ we construct a natural transformation $\theta : F_y \to F_x$ that is, at each object z, a linear map $\theta_z : F_y(z) \to F_x(z)$, specified by $\theta_z(\phi) = F_x(\phi)(\alpha) = \phi\alpha = \alpha^*(\phi)$. This is because the definition of the functor $F_x = \operatorname{Hom}(x, -)$ on morphisms is by composition of morphisms.

2. is 6.3 of ASS and 3. is 6.4. $\square$

Corollary 7.0.8. *Let A be a finite dimensional algebra and U an A-module.*

1. *The projective functor F_U has endomorphism ring isomorphic to $\operatorname{End}_A(U)$*

2. *F_U is indecomposable if and only if U is indecomposable.*

Definition 7.0.9. Assem-Simson-Skowronski choose to define a functor $M \in \operatorname{Fun} A$ to be *finitely generated* if and only if there is a finitely generated A-module U so that M is a quotient of F_U.

It does not seem obvious that this has much to do with the concept of finite generation we know in module theory. To make this connection we make the following definition. Given a functor $M : \mathcal{C} \to K$-mod and elements $a_i \in M(x_i)$, where the x_i are various objects of $\mathcal{C}$ and i lies in some indexing set I, then $\langle a_i \mid i \in I \rangle$ is the smallest subfunctor of M that contains the a_i. We say that M is generated by the a_i if $M = \langle a_i \mid i \in I \rangle$.

Proposition 7.0.10. *F_x is generated by $1_x \in F_x(x)$ or, indeed, by any automorphism in $F_x(x)$.*

Class Activity. Is the last proposition obvious? If A is a ring with an idempotent $e \in A$, is it obvious that the projective module Ae is generated by e?

Proposition 7.0.11. *The following are equivalent for a functor $M \in \operatorname{Fun} A$.*

1. *There is a finitely generated A-module U so that M is a quotient of F_U.*

2. *There is a finitely generated A-module U and an element $b \in M(U)$ so that $M = \langle b \rangle$.*

3. *There are finitely many finitely generated A-modules U_i and elements $a_i \in M(U_i)$ so that M is generated by the a_i.*

Class Activity. How much of this is obvious?

Corollary 7.0.12. *The F_U, for finitely generated A-modules U, provide a complete list of finitely-generated projective objects in* Fun A. *The projective F_U is indecomposable if and only if the module U is indecomposable.*

Proof. This is 6.5 of Assem-Simson-Skowronski. □

Theorem 7.0.13. *Let U be an indecomposable A-module.*

1. *The functor $\mathrm{Rad}_A(U, -)$ is the unique maximal subfunctor of F_U. The quotient $S_U = F_U / \mathrm{Rad}_A(U, -)$ is the unique simple quotient of F_U.*

2. *The S_U form a complete list of simple functors in* Fun A.

3. *F_U is the projective cover of S_U.*

Proof. See 6.6 - 6.8 of ASS.

1. We have seen that for each indecomposable A-module V we have $\mathrm{Rad}_A(U, V) = F_U(V)$ if $V \not\cong U$, and if $V = U$ then $F_U(U) - \mathrm{Rad}_A(U, U)$ consists of the isomorphisms $U \to U$. Thus if M is a subfunctor of F_U that is not contained in $\mathrm{Rad}_A(U, -)$ then $M(U)$ contains an isomorphism of U. Such an isomorphism generates F_U, so this means $M = F_U$.

2. If T is a simple functor then $T(U) \neq 0$ for some indecomposable module U. Because T is simple it is generated by any non-zero element $a \in T(U)$, and the unique homomorphism $F_U \to T$ determined by a is surjective. Because F_U has S_U as its unique simple quotient, $T \cong S_U$.

3. The fact that the quotient map is essential is the same as a homework exercise. □

We now do 6.9, 6.10, 6.11 of Assem-Simson-Skowronski.

Proposition 7.0.14. *Let $\alpha : U \to V$ be a morphism of A-modules with U indecomposable.*

1. *α is left almost split if and only if the image of $\alpha^* : F_V \to F_U$ is $\mathrm{Rad}_A(F_U)$; if and only if $\mathrm{Coker}\,\alpha^* \cong S_U$*

2. *α is left minimal if and only if $\alpha^* : F_V \to F_U$ is right minimal; if and only if $F_V \to \mathrm{Im}\,\alpha^*$ is minimal; if and only if $F_V \to \mathrm{Im}\,\alpha^*$ is a projective cover.*

Proof. 1. For any homomorphism $\theta : U \to X$ there exists $\gamma : V \to X$ with $\theta = \alpha\gamma$ if and only if θ lies in the image of $\alpha_X^* : \mathrm{Hom}_A(V, X) \to \mathrm{Hom}_A(U, X)$. Thus α is left almost split if and only if, whenever X is indecomposable, the image of α_X^* consists of the non-isomorphisms. This happens if and only if the image of α^* is $\mathrm{Rad}(F_U)$.

2. To prove the first implication '$\Rightarrow$', suppose that α is left minimal and suppose that $\alpha^* = \alpha^*\theta$ for some $\theta \in \mathrm{End}(F_V) \cong \mathrm{End}(V)$. Then $\theta = \beta^*$ for some $\beta \in \mathrm{End}(V)$ and $\alpha^* = \alpha^*\beta^* = (\beta\alpha)^*$. We deduce from the bijection of morphisms that $\alpha = \beta\alpha$. Thus β is an isomorphism, and so is β^*.

We now prove the first '$\Leftarrow$'. Suppose that α^* is right minimal and $\alpha = h\alpha$ for some h. Then $\alpha^* = \alpha^* h^*$ so h^* is an isomorphism, hence so is β^*.

Writing $\bar{\alpha}^* : F_V \to \text{Im}\,\alpha^*$ for the map determined by α^*, it is evident that there exists θ with $\alpha^* = \alpha^* \theta$ if and only if there exists θ with $\bar{\alpha}^* = \bar{\alpha}^* \theta$, and this proves the middle equivalence.

For the final '$\Rightarrow$', suppose that $\bar{\alpha}^*$ is right minimal. We show that it is essential epi, and for this we only need to show that it is essential. Suppose that $M \subset F_V$ has the property that $\bar{\alpha}^*(M) = \bar{\alpha}^*(F_V)$. By projectivity of F_V there exists $\phi : F_V \to M$ so that $\bar{\alpha}^* = \phi(\bar{\alpha}^*)|_M$, and if $\bar{\phi}$ is the composite of ϕ and inclusion $M \subset F_V$ then $\bar{\alpha}^* = \bar{\phi}\bar{\alpha}^*$ thus $\bar{\phi}$ is an isomorphism and this forces $M = F_U$. We leave the implication '$\Leftarrow$' as an exercise. $\square$

Theorem 7.0.15. *1. Let $0 \to U \to V \to W \to 0$ be a short exact sequence of A-modules. It is an almost split sequence if and only if U is indecomposable and*

$$0 \to F_W \to F_V \to F_U \to S_U \to 0$$

is a minimal projective resolution.

2. Let U be indecomposable and injective. Then

$$0 \to F_{U/\operatorname{Soc} U} \to F_U \to S_U \to 0$$

is a minimal projective resolution.

Proof. 1. Observe that $0 \to F_W \to F_V \to F_U$ is necessarily exact, because $\text{Hom}(-, X)$ is always left exact. The rest follows from the preceding proposition and the observation that if the second sequence is exact then the first cannot be split.

2. If X is indecomposable then any non-isomorphism $\theta : U \to X$ has $\operatorname{Soc} U$ in its kernel, because otherwise U would embed in X and be a direct summand. This means that θ factors through $U/\operatorname{Soc} U$ and so lies in the image of $F_{U/\operatorname{Soc} U} \to F_U$, which therefore contains $\text{Rad}(F_U)$. This map cannot be surjective because F_U is projective. Therefore the sequence is exact in the middle and at the right. We get exactness at the left because Hom is left exact. $\square$

Corollary 7.0.16. *The simple functors in $\text{Fun}A$ are finitely presented and have projective dimension at most 2. When A has finite representation type, $\text{Fun}A$ and the Auslander algebra of A have global dimension at most 2.*

Corollary 7.0.17. *Let U be a finite dimensional A-module. Then $\text{Rad}(F_U)$ is finitely generated.*

Corollary 7.0.18. *Let $0 \to U \to V_1 \oplus \cdots \oplus V_n \to W \to 0$ be an almost split sequence where $V_1, \ldots, V_n$ are indecomposable. Then $\text{Rad}(F_U)/\text{Rad}^2(F_U) \cong S_{V_1} \oplus \cdots \oplus S_{V_n}$. We deduce from this that for each indecomposable module V, the multiplicity of V as a summand of the middle term of the almost split sequence starting at U is $\dim_{D_V} \text{Irr}(U, V)$*

Exercise 7.0.19. Characterize the kinds of indecomposable modules U for which the simple functor S_U has projective dimension 2, 1 and 0.

Chapter 8

Torsion pairs and tilting

Example: Morita theory.

Chapter 9

The bounded derived category

Triangulated categories; Auslander-Reiten triangles; tilting complexes and derived equivalence.

Chapter 10

Bibliography

[1] I. Assem, D. Simson and A. Skowronski, *Elements of the representation theory of associative algebras Vol. 1: Techniques of representation theory,* London Mathematical Society Student Texts **65**, Cambridge University Press, Cambridge, 2006.

[2] M. Auslander, I. Reiten and S.O. Smalø, *Representation theory of Artin algebras,* Cambridge studies in advanced mathematics **36**, Cambridge University Press 1995.

[3] A.I. Generalov, *Relative homological algebra, cohomology of categories, posets and coalgebras,* pp. 611–638 in M. Hazewinkel (ed.), Handbook of Algebra, Vol. I, Elsevier 1996.

[4] D. Happel, *Triangulated Categories in the Representation Theory of Finite-Dimensional Algebras,* London Math. Soc. Lecture Note Ser. **119**, Cambridge University Press, Cambridge, 1988.

[5] S. MacLane, *Categories for the Working Mathematician,* Grad. Texts in Math. **5**, Springer-Verlag, New York, 1971.

[6] B. Mitchell, *Rings with several objects,* Advances in Math. **8** (1972), 1–161.

[7] P.J. Webb, *An introduction to the representations and cohomology of categories,* pp. 149–173 in: M. Geck, D. Testerman and J. Thévenaz (eds.), Group Representation Theory, EPFL Press (Lausanne) (2007).