Math3592 review sheet for the final exam

The exam is on Thursday 12/16/10 from 1:30-4:30 in room Molecular Cellular Biology 2-120. I will hold a review session in our usual lecture room from 10:00-12:00 on Thursday morning.

There are 12 questions on the exam, some divided into parts, with each question part usually worth 6% of the total. You may not use books or notes. You may use a calculator. Always show your work, and be sure to write down sufficient detail so that I can see that you are able to do all calculations without a calculator if necessary. If you are not sure what is required in any question, or what the question means, do ask.

1. Let $f : \text{Mat}(2, 2) \to \mathbb{R}$ be the mapping $f(A) = \text{trace}(A^2)$. Find the directional derivative of $f$ at the matrix $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ in the direction of the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

2. Consider the following functions which are defined to be 0 at $(0, 0)$. Are they continuous at the origin? Differentiable? Do the partial derivatives exist?

   $\frac{x^2}{\sqrt{x^2 + y^2}}, \quad \frac{2x - 5y}{\sqrt{x^2 + y^2}}, \quad \frac{xy}{\sqrt{x^2 + y^2}}, \quad \frac{x^2 + y^2}{x + y^2}$

3. True or false? For each of the following statements, decide whether it is true or false, and then either give brief reasons or a counterexample to justify your assertion.

   (a) There exists a surjective linear mapping $\mathbb{R}^7 \to \mathbb{R}^{10}$.

   (b) If $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function, there can never be a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ with $gf = 1_{\mathbb{R}^2}$, the identity mapping on $\mathbb{R}^2$.

   (c) If $f : \mathbb{R}^2 \to \mathbb{R}^3$ is a differentiable function, there can never be a function $g : \mathbb{R}^3 \to \mathbb{R}^2$ with $fg = 1_{\mathbb{R}^3}$, the identity mapping on $\mathbb{R}^3$.

   (d) Let $v_1, \ldots, v_r$ be a linearly independent set of vectors in a vector space $V$ and $w_1, \ldots, w_r$ another set of vectors in a vector space $W$. Then there exists a linear mapping $T : V \to W$ with $T(v_i) = w_i$ for all $i$ with $1 \leq i \leq r$.

   (e) If $S$ is an $m \times n$ matrix of rank $m$ then there exists an $n \times m$ matrix $T$ with $ST = I$, the identity matrix.

   (f) Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is continuously differentiable and there exists $g : \mathbb{R}^n \to \mathbb{R}^n$ with $fg = gf = 1$. Then $g$ is differentiable.

   (g) If $f : U \to V$ and $g : V \to W$ are linear mappings then $\text{rank}(gf) \leq \text{rank}(f)$ always.

   (h) If $S : U \to V$ is a linear mapping which is onto then there exists a linear mapping $T : V \to U$ with $ST = I$.

   (i) If $S : U \to V$ is a linear mapping which is onto then there exists a linear mapping $T : V \to U$ with $TS = I$. 

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(j) If \( S : U \to V \) is a linear mapping which is 1-1 then there exists a linear mapping \( T : V \to U \) with \( ST = I \).

(k) If \( S : U \to V \) is a linear mapping which is 1-1 then there exists a linear mapping \( T : V \to U \) with \( TS = I \).

4. Find the number of paths of length 4 from vertex A to itself in the graph

5. Let \( S \) be a subset of \( \mathbb{R}^n \). We will say that \( x \) is a limit point of \( S \) \( \iff \) for all \( \epsilon > 0 \) there exists \( y \in S \) with \( 0 < |x - y| < \epsilon \). Using the definition that \( S \) is closed \( \iff \) for every point \( x \) not in \( S \) there is a ball of some positive radius with center \( x \) which contains no point of \( S \), prove that

\[
S \text { is closed } \iff S \supseteq \text{its limit points}.
\]

Which of the following statements means \( x \) is not a limit point of \( S \)?

(i) There exists \( \epsilon > 0 \) such that for all \( y \in S \) either \( y = x \) or \( |y - x| \geq \epsilon \).

(ii) There exists \( \epsilon > 0 \) such that for all \( y \in S \) either \( y = x \) or \( |y - x| > \epsilon \).

(iii) There exists \( \epsilon > 0 \) such that there exists \( y \in S \) with either \( y = x \) or \( |y - x| \geq \epsilon \).

(iv) There exists \( y \in S \) such that there exists \( \epsilon > 0 \) with either \( y = x \) or \( |y - x| > \epsilon \).

6. Do one step of Newton’s method to solve the system of equations

\[
\begin{align*}
ye^x + xe^y &= 1 \\
x^3 + xy + \sin y &= 0
\end{align*}
\]

starting at \( a_0 = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \).

7. Calculate \( \det \begin{pmatrix} 1 & 2 & -1 \\ 5 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix} \).

8. Prove that

\[
Df(a)(h) = \lim_{t \to 0} \frac{f(a + th) - f(a)}{t}.
\]

9. Let \( T : \mathbb{R}^3 \to \mathbb{R}^2 \) be the linear map whose matrix with respect to the standard bases is \( \begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix} \). Find the matrix of \( T \) with respect to the bases

\[
\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \begin{pmatrix} 3 \\ -2 \\ 1 \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ 1 \end{pmatrix} \quad \text{and} \quad \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \begin{pmatrix} 1 \\ 1 \end{pmatrix}.
\]